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theoretical mechanics

an introduction to mathematical physics

Joseph Sweetman Ames
and Francis D. Murnaghan

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PREFACE

Owing to the notable advances in mathematical physics in recent years, due in large part to the introduction of the quantum principle and the principles of relativity, there has been a great stimulus in the study of theoretical mechanics. Many new methods have been introduced; a more critical discussion of the underlying principles has been made; a new emphasis has been placed upon various features of the subject. Furthermore, new mathematical processes have been developed. It is evident, therefore, that students of physics, mathematics, and chemistry should have at their disposal a textbook or book of reference corresponding to the new point of view. It is the purpose of this book to meet this demand.

It is realized that there are many students who are not associated with institutions where advanced instruction is given and who feel the need of this new knowledge concerning theoretical mechanics. Consequently this book has been so written that it may be read by any competent student without the aid of an instructor, and numerous problems have been introduced. If the book is used with a class, the instructor may devote his time to extending the subject or to the discussion of new applications. In pursuit of this same purpose, an attempt is made to give rigorous proofs of theorems and formulas, so that the student, if he is not proficient in mathematics, may obtain a clearer idea of what is needed in a mathematical science.

It is probable that in the first reading of the book a student may find it desirable to omit certain sections, or, possibly, chapters, but he should bear in mind that if he wishes to proceed to the serious study of any of the more advanced fields of modern physics, such as statistical mechanics, quantum dynamics, atomic theory, etc., there is no chapter which will not prove of use. Furthermore, references are given to books and articles in which the student will find certain subjects amplified.

J. S. A.
F. D. M.

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THEORETICAL MECHANICS

CHAPTER I

VECTOR ANALYSIS

1. The space-time postulates. It is necessary in theoretical mechanics to postulate certain properties of space and time so that numbers may be assigned to elements of length, area, and volume and to intervals of time. The fundamental postulates involved in the processes of measurement will be discussed in a later chapter. Classical mechanics assumes that space is of three dimensions and possesses the properties postulated by Euclid. Among these attention may be specifically directed to the postulate that any figure in space may be moved freely about, and to what is known as the parallel axiom. This latter is essentially the same as a postulate of the existence of figures of any magnitude similar to a given figure. The significance of these postulates is grasped most readily by considering figures of two dimensions — that is, surfaces — and comparing the plane with the sphere and the ellipsoid. The plane is Euclidean. Figures on it may be freely moved about and enlarged or diminished arbitrarily in scale. Briefly, there exist both congruent and similar figures on the plane.. On the sphere a figure may be moved about freely; but the postulate of similarity is not satisfied, that is, the shape of a spherical figure is not independent of its magnitude. The sphere is an example of what is known as a space of constant curvature. In such spaces the postulate of congruence is obeyed. On the other hand, neither the postulate of congruence nor that of similarity is obeyed on the ellipsoid. A figure on an ellipsoid cannot be moved about without distortion. An ellipsoid is an example of a space of variable curvature. For such spaces the postulate of congruence is not obeyed.

As regards time, classical mechanics postulates a definite method of assigning numbers to intervals of time, based, as a rule, on the definition of the word "periodic" in terms of the vibrations of a pendulum or the motion of some convenient mechanical system.

It is postulated — although this is not always explicitly stated — that the dimensions of a space figure have the same numeric values for all observers, whether the latter are moving with the figure or not; that is, the dimensions of a body are supposed to be *absolute* properties of the body and not merely the expression of a relation between the body and the observer. In short, the observer (the man who makes experiments) is not mentioned in classical mechanics. In the modern theory of relativity the point of view is quite different.

2. The vector concept. In order to locate a point in a Euclidean space of three dimensions it is necessary to have a rigid material framework, with respect to which measurements can be made. This material framework is known as a reference frame or, briefly, a frame. It is usually most convenient to use as a frame a system of three mutually perpendicular axes, $OXYZ$, meeting in a point O known as the origin. We shall adopt the convention that these are so lettered as to form a right-handed system; that is, the rotation of a right-handed screw whose axis coincides with OZ from the x to the y axis forces the point of the screw to advance along the z axis. For example, O might be the center of the earth, with OZ pointing toward the north pole. Then OX and OY would be any two perpendicular lines through O in the equatorial plane, with OY east of OX. The position of any point M is, then, identified by means of its Cartesian coördinates (x, y, z) relative to the frame $OXYZ$, as shown in Fig. 1.

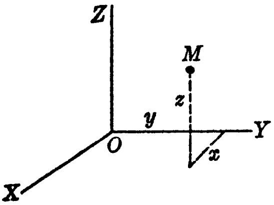

FIG. 1

We are thus able, in particular, to identify a new reference frame $O'X'Y'Z'$, and any point M may, if convenient, be identified by means of its Cartesian coördinates (x', y', z') relative to the new frame $O'X'Y'Z'$. Let us denote the coördinates of the new origin O' relative to the original frame by (x_0, y_0, z_0), and let the relative orientation of the two frames be specified by means of the table of direction cosines

	OX	OY	OZ
$O'X'$	a_1	b_1	c_1
$O'Y'$	a_2	b_2	c_2
$O'Z'$	a_3	b_3	c_3

where c_2, for example, is the cosine of the angle between the OZ and the $O'Y'$ axes. We have, then, the relations

$$a_1^2 + b_1^2 + c_1^2 = 1, \text{ etc.,} \quad a_2a_3 + b_2b_3 + c_2c_3 = 0, \text{ etc.,} \tag{2.1}$$

expressing that (a_1, b_1, c_1) etc. are the direction cosines of the axis $O'X'$ relative to $OXYZ$, etc., and that the lines $O'Y'$ and $O'Z'$ are at right angles, etc. Upon interchanging the rôles of the two frames, we have the relations

$$a_1^2 + a_2^2 + a_3^2 = 1, \text{ etc.,} \quad b_1c_1 + b_2c_2 + b_3c_3 = 0, \text{ etc.} \tag{2.2}$$

The connection between the coördinates (x, y, z) and (x', y', z') of the same point M, relative to the two frames $OXYZ$ and $O'X'Y'Z'$ respectively, may be given conveniently by the table

	$x - x_0$	$y - y_0$	$z - z_0$
x'	a_1	b_1	c_1
y'	a_2	b_2	c_2
z'	a_3	b_3	c_3

which is read as follows:

$$\begin{aligned} x' &= a_1(x - x_0) + b_1(y - y_0) + c_1(z - z_0), \text{ etc.,} \\ x - x_0 &= a_1x' + a_2y' + a_3z', \text{ etc.} \end{aligned} \tag{2.3}$$

When we consider two points, M_1 and M_2, we may introduce the idea of the operation which *carries* us from one of the points, M_1, to the other point, M_2. This step from M_1 to M_2 contains, in addition to the quality of magnitude, that is, the distance between the two points, the quality of "direction."

There are many physical quantities which have exactly these two characteristics, and these may be conveniently represented by the geometric operation described above. Such physical quantities are known as *vector* quantities. A vector quantity is sufficiently described when its magnitude and direction are given. One of the two end points, M_1 and M_2, of the segment which is used to represent the vector quantity may be chosen arbitrarily; the other is then definitely determined by the magnitude and direction of the representative segment. Calling the segment itself a vector, we may say that a vector quantity may be represented by any one of a system of equal, parallel, and similarly directed vectors. In what follows, the term "vector" will be used to denote either the physical vector quantity or a geometric segment representative of the vector quantity.

The magnitude and direction of the step from M_1 to M_2 may be specified by means of the projections of the segment M_1M_2 on the coördinate axes. These projections will, in general, be different in different reference frames; and we shall say that the projections furnish, *in the given reference frame*, a *presentation* of the segment, so that we shall have in each frame a *different* presentation of the *same* segment. Thus, if (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coördinates, respectively, of M_1 and M_2 relative to $OXYZ$, the presentation of the segment M_1M_2 in the $OXYZ$ frame is $(x_2-x_1, y_2-y_1, z_2-z_1)$. Similarly, the presentation in the $O'X'Y'Z'$ frame is $(x'_2-x'_1, y'_2-y'_1, z'_2-z'_1)$, and, as follows readily from (2.3), the two presentations are connected with one another by the equations

$$\begin{aligned}
x'_2 - x'_1 &= a_1(x_2 - x_1) + b_1(y_2 - y_1) + c_1(z_2 - z_1),\\
y'_2 - y'_1 &= a_2(x_2 - x_1) + b_2(y_2 - y_1) + c_2(z_2 - z_1),\\
z'_2 - z'_1 &= a_3(x_2 - x_1) + b_3(y_2 - y_1) + c_3(z_2 - z_1),\\
x_2 - x_1 &= a_1(x'_2 - x'_1) + a_2(y'_2 - y'_1) + a_3(z'_2 - z'_1),\\
y_2 - y_1 &= b_1(x'_2 - x'_1) + b_2(y'_2 - y'_1) + b_3(z'_2 - z'_1),\\
z_2 - z_1 &= c_1(x'_2 - x'_1) + c_2(y'_2 - y'_1) + c_3(z'_2 - z'_1).
\end{aligned} \tag{2.4}$$

The question then arises, How can we be sure that a given physical quantity is a vector quantity? In other words, precisely what is meant by the statement that a certain physical quantity has the quality of direction? Let us suppose that we have a physical process which involves the measurement of three quantities, and which furnishes us, therefore, with three numbers for each reference frame. Can we regard these three numbers as the presentation of a definite vector in the corresponding frame? Let us express the query in a different form. Let (A_x, A_y, A_z) be the three numbers associated with the frame $OXYZ$, and let (A'_x, A'_y, A'_z) be those associated, for the same physical process, with the frame $O'X'Y'Z'$. Will the segment defined by choosing the former set as the measures of its projections on the OX, OY, and OZ axes, respectively, have the numbers (A'_x, A'_y, A'_z) as the measures of its projections on the $O'X'$, $O'Y'$, and $O'Z'$ axes respectively? If this is so, the physical process in which we are interested defines a vector quantity, because the physical quantity furnished to us is fully described by a representative segment. The same observations apply to any mathematical process which furnishes us a set of three quantities associated with each reference frame. The test is that (A'_x, A'_y, A'_z) must measure the projections on the axes of the $O'X'Y'Z'$ frame of a segment whose projections on the axes of the $OXYZ$ frame are measured by (A_x, A_y, A_z). If M_1M_2 denotes any such segment, we may say that

$$A_x = x_2 - x_1, \quad A_y = y_2 - y_1, \quad A_z = z_2 - z_1;$$

and if we have a vector quantity, it must follow that

$$A'_x = x'_2 - x'_1, \quad A'_y = y'_2 - y'_1, \quad A'_z = z'_2 - z'_1.$$

We have, then, from (2.4), the equations

$$\begin{aligned} A'_x &= a_1A_x + b_1A_y + c_1A_z, & A_x &= a_1A'_x + a_2A'_y + a_3A'_z, \\ A'_y &= a_2A_x + b_2A_y + c_2A_z, & A_y &= b_1A'_x + b_2A'_y + b_3A'_z, \\ A'_z &= a_3A_x + b_3A_y + c_3A_z, & A_z &= c_1A'_x + c_2A'_y + c_3A'_z. \end{aligned} \tag{2.5}$$

These equations enable us to answer the question as to whether the sets of three numbers with which we are furnished, one set to each frame, present a vector or not. It is, of course, possible to regard an *arbitrary* set of three numbers (A_x, A_y, A_z) as presenting a vector in the frame $OXYZ$. The presentation (A'_x, A'_y, A'_z) in any other frame is then furnished by the equations (2.5). But this is not the usual situation. We are, in general, in possession of a process furnishing us with a set of three numbers in each frame, so that the three numbers in any reference frame are known. The equations

(2.5) then tell us whether or not these sets of three numbers present a vector or not. If they do present a vector, we shall denote this vector by the symbol **A**. This symbol denotes the *vector* itself and not any particular presentation of it. Or, if we wish, **A** stands for all possible presentations of the vector, any two of the presentations, by means of Cartesian frames, being connected with each other by the equations (2.5). The numbers (A_x, A_y, A_z) will be called the components of the vector **A** in the frame $OXYZ$.

The properties and combinations of vectors which are important are those that have no particular reference to the presentation of the vectors which it is convenient to adopt. Thus the common length of the representative segments M_1M_2 of a vector **A** is independent of the particular frame in use. In the frame $OXYZ$ it has the value $(A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}}$, and it is easy to verify, from the equations (2.5) and (2.2), that $A'^2_x + A'^2_y + A'^2_z = A_x^2 + A_y^2 + A_z^2$. We shall denote the length of any representative segment of the vector **A** by A and shall call it the *magnitude* of **A**. A quantity such as A has a value quite independent of the particular frame in use, and is known as an *invariant* or *scalar* quantity.

As an example of a relationship between two vectors which is independent of the frame, we may consider the equality of two vectors. Two vectors, **A** and **B**, are said to be equal when they have the same representative segments, that is, when any representative segment of the first is equal, parallel, and similarly directed to any representative segment of the second. This implies that $A_x = B_x$, $A_y = B_y$, $A_z = B_z$; and it follows from (2.5) that these equations imply that $A'_x = B'_x$, etc. Thus, if two vectors have the same presentation in any one frame, they have the same presentation in any other frame. The two vectors may then be said to be equal. If **A** and **B** are furnished *a priori*, it would be more natural to say that they are the same vector than to say that they are equal vectors; but when **A** and **B** result from certain processes performed upon other quantities, — that is, when **A** and **B** are secondary or derived vectors rather than primary or given vectors, — the latter terminology is more convenient.

3. Vector algebra. *a. Multiplication of a vector by a scalar.* Let **A** be an arbitrary vector whose presentation in the frame

$OXYZ$ is (A_x, A_y, A_z), and let m be the measure of any scalar quantity. If we multiply each of the components (A_x, A_y, A_z) by m, we have a process which furnishes us with a set of three numbers in each frame. In the frame $OXYZ$ the set is (mA_x, mA_y, mA_z), and in any other frame, $O'X'Y'Z'$, the set is (mA'_x, mA'_y, mA'_z). It follows at once, from (2.5), that these sets of three numbers present a vector. If m is different from zero, any representative segment of the new vector is parallel to any representative segment of **A**, the two segments being similarly directed if m is positive, and oppositely directed if m is negative. The ratio of the length of the representative segment of the new vector to the length of the representative segment of **A** is the numeric value of m. The new vector is denoted by $m\mathbf{A}$ and is said to be the product of the vector **A** by the scalar m. Its magnitude is $|m|A$, where $|m|$ is the numeric value of m.

When $m = 0$ we have an exceptional situation. The presentation of the new vector is (0, 0, 0) and is quite independent of the original vector **A**. There is no representative segment, so that the new vector cannot properly be said to possess the quality of direction. Since, when m is different from zero and positive, $m\mathbf{A}$ has the direction of **A**, and since, when m is zero, the new vector is independent of the original vector **A**, we say that its direction is indeterminate. This vector, whose components in any frame are (0, 0, 0), is called the zero vector. Its components in any other frame are also (0, 0, 0), so that it has the same presentation in all reference frames. We shall denote the zero vector by the symbol **0**. It is apparent that its introduction is to a certain extent an enlargement of the vector concept. All other vectors have a definite direction, while the zero vector has no determinate direction. Once it is introduced, however, we are enabled to multiply a vector by any scalar quantity without having to guard against the possibility of the scalar taking the value zero.

b. The addition and subtraction of vectors. From any vector **A** we derive, on multiplication by the scalars m, n, and $m + n$, the three vectors $m\mathbf{A}$, $n\mathbf{A}$, and $(m + n)\mathbf{A}$ respectively. If the initial point of a representative segment of $n\mathbf{A}$ is chosen to coincide with the end point of a representative segment of $m\mathbf{A}$, the segment from the initial point of the representative segment of

$m\mathbf{A}$ to the end point of the representative segment of $n\mathbf{A}$ is a representative segment of $(m+n)\mathbf{A}$. We shall write

$$m\mathbf{A} + n\mathbf{A} = (m+n)\mathbf{A},$$

thus defining the process of adding (or subtracting) two vectors having the same direction. This definition of addition makes the distributive law applicable to the multiplication of a vector by a scalar. The addition of equal and oppositely directed vectors (or, equivalently, the subtraction of equal vectors) gives the zero vector.

We shall now consider two vectors, **A** and **B**, whose representative segments are not necessarily parallel. Let their presentations in any frame $OXYZ$ be (A_x, A_y, A_z) and (B_x, B_y, B_z), respectively, and form the algebraic sums $(A_x + B_x, A_y + B_y, A_z + B_z)$. We have here again a process furnishing us in each frame with a set of three numbers, the set in any other frame $O'X'Y'Z'$ being $(A'_x + B'_x, A'_y + B'_y, A'_z + B'_z)$. It follows at once from the equations (2.5) that these sets of three numbers present a vector, which we shall denote by $\mathbf{A} + \mathbf{B}$. A representative segment of $\mathbf{A} + \mathbf{B}$ can be found as follows:

*Choose the initial point of a representative segment of **B** as coinciding with the end point of a representative segment of **A**. Then the segment from the initial point of the representative segment of **A** to the end point of the representative segment of **B** is a representative segment of **A** + **B**.*

Since
$$B_x + A_x = A_x + B_x, \quad B_y + A_y = A_y + B_y,$$
$$B_z + A_z = A_z + B_z,$$

it follows that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. Thus addition of vectors is commutative, and we may call $\mathbf{A} + \mathbf{B}$ the sum of **A** and **B**, it being immaterial which vector is taken first in forming the sum. Vector addition is very similar to scalar addition. Thus, if we have a series of numbers on a scale, they are added or subtracted according to the rule explained for vectors whose representative segments are parallel. The only extension necessary for vectors whose representative segments are not parallel is that, in moving a representative segment of one of the vectors so that its initial point may coincide with the end point of a representative segment of the other vector, we must be careful to preserve the direction of the segment which is moved. (See Fig. 2.)

Subtraction of vectors is defined by the statement

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

To obtain a representative segment of **A** − **B**, we must first reverse the direction of the representative segment of **B** and then proceed as in addition. It follows that representative segments of **A** + **B** and **A** − **B** are given by the two diagonals of the parallelogram whose coterminous edges are representative segments of **A** and **B** respectively.

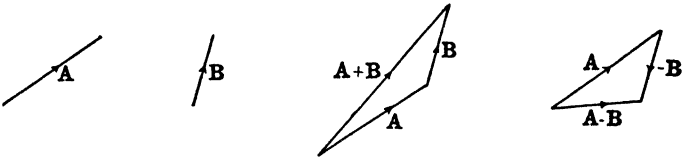

FIG. 2

In connection with the definition of the sum of two vectors it should be pointed out that there are certain quantities in physics which may be represented by segments of directed lines, but which are of such a character that the combined effect of several of them is not found by adding the separate representative segments. As an example, let us consider the rotation of a rigid body, about a line OP as an axis, through a definite angle θ. The displacement of each point of the rigid body is fully described by a directed line of definite length if the line is drawn along the axis, if its length is made equal to θ, and if the direction of the line is so chosen as to indicate by some rule, such as the right-hand-screw convention, the sense of the rotation. Now let us suppose that a rigid body is rotated around two lines, OP_1 and OP_2, in succession, through angles equal to the lengths of OP_1 and OP_2 respectively. The net result of these two rotations is equivalent to a rotation about a line through O, but this line is not the diagonal of the parallelogram whose coterminous edges are OP_1 and OP_2. (See § 24.)

In order to avoid confusion, such a directed quantity is not called a vector quantity, even though it can be represented by a segment and has presentations in different frames which are connected with one another by the equations (2.5); for we have a preconceived idea as to what we should mean by the sum of two rotations about a point, and this idea does not agree with the definition of the sum of two vectors which is most generally convenient. The same question as to the suitability of the definition of "sum" will arise, naturally, in connection with every directed physical quantity. If, however, the directed quantity is arrived at as the result of operations upon some primary vector quantities, we usually know beforehand, by the rules of vector algebra and analysis, whether or not it is a vector in the narrow sense, that is, whether or not it obeys the rule of addition. Thus, in the case of the rotation of the rigid body about a point the primary vectors are the displacements of the various points of the rigid body. The operation of deducing from these displacement vectors the nature of the segment which is representative of the rotation is not linear; hence the finite rotations of a rigid body about a point are not vectors in the narrow sense of the term.

It is well to emphasize the distinction between experimental and theoretical physics. It is by experiment in the former and by postulation in the latter that the result of compounding physical quantities is determined. In the case of any physical quantity two questions arise: (1) Is it a vector in the wider sense; that is, is it fully described by a magnitude and a direction? (2) Is the experimental law for compounding the physical quantity the same as that for vectors? The answer to (2) may be in the negative even though the answer to (1) is in the affirmative.

A vector whose magnitude is unity is called a unit vector. Associated with any frame $OXYZ$ there are three fundamental unit vectors whose representative segments lie along the three coördinate axes respectively. These unit vectors have presentations (1, 0, 0), (0, 1, 0), (0, 0, 1) in the $OXYZ$ frame and are denoted, respectively, by **i**, **j**, **k**. We have the equation

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},$$

in which **A** is an arbitrary vector whose presentation in the frame $OXYZ$ is (A_x, A_y, A_z).

c. The scalar product of two vectors. As the direction cosines, relative to a frame $OXYZ$, of any representative segments of two vectors **A** and **B** are $\left(\frac{A_x}{A}, \frac{A_y}{A}, \frac{A_z}{A}\right)$ and $\left(\frac{B_x}{B}, \frac{B_y}{B}, \frac{B_z}{B}\right)$, respectively, the angle θ between these segments has its cosine given by the formula $\cos\theta = (A_xB_x + A_yB_y + A_zB_z) \div AB$. Hence, since the expression

$$A_xB_x + A_yB_y + A_zB_z$$

is equal to $AB\cos\theta$, it has a value independent of the frame in which the vectors are presented. We have here a process furnishing us with a number in each reference frame, and this number is the same for all frames. It is easy to verify, by means of the equations (2.5) and (2.2), that

$$A'_xB'_x + A'_yB'_y + A'_zB'_z = A_xB_x + A_yB_y + A_zB_z.$$

This number is called the scalar product of the vector **A** and the vector **B** and is denoted by $(\mathbf{A}.\mathbf{B})$, so that we have the equation of definition

$$(\mathbf{A}.\mathbf{B}) = A_xB_x + A_yB_y + A_zB_z = AB\cos\theta. \tag{3.1}$$

It follows from (3.1) that $(\mathbf{A}.\mathbf{B}) = (\mathbf{B}.\mathbf{A})$, so that the scalar product of two vectors obeys the commutative law. It obeys also the distributive law, since

$$\begin{aligned}(\mathbf{A}.\mathbf{B}+\mathbf{C}) &= A_x(B_x + C_x) + A_y(B_y + C_y) + A_z(B_z + C_z)\\ &= (A_xB_x + A_yB_y + A_zB_z) + (A_xC_x + A_yC_y + A_zC_z)\\ &= (\mathbf{A}.\mathbf{B}) + (\mathbf{A}.\mathbf{C}).\end{aligned}$$

The equation $(\mathbf{A}.\mathbf{B}) = AB\cos\theta$ shows that when the scalar product of two vectors is zero, then one of the vectors is zero or else the vectors have perpendicular representative segments. Since the zero vector has no determinate direction, it may be regarded as perpendicular to any vector, and we have the result that the necessary and sufficient condition for two vectors to be at right angles (that is, to have perpendicular representative segments) is that their scalar product be zero.

If the vector **A** is a unit vector, $(\mathbf{A}.\mathbf{B}) = B\cos\theta$ is called the resolved part of **B** in the direction of **A**. This resolved part has a maximum value B when the direction of **A** is the same as that of **B**, and a minimum value $-B$ when the direction of **A** is opposite to that of **B**. (B_x, B_y, B_z) are, respectively, the resolved parts of **B** in the directions of the coördinate axes of the frame $OXYZ$.

Let us now suppose that we have a process furnishing three numbers (A_x, A_y, A_z) in each frame $OXYZ$, and let us suppose that these three numbers have the property that the expression $A_xB_x + A_yB_y + A_zB_z$ has a value independent of the frame. Here (B_x, B_y, B_z) is the presentation in the frame $OXYZ$ of an arbitrary vector **B**. It then follows that (A_x, A_y, A_z) present, in the frame $OXYZ$, a vector **A**. To see this we have merely to take, for example, as the frame $O'X'Y'Z'$ that in which the vector **B**, which we may assume, without loss of generality, to be a unit vector, has the presentation (1, 0, 0). Its presentation in the $OXYZ$ frame is, then, from the equations (2.5), (a_1, b_1, c_1); and the given equality

$$A'_xB'_x + A'_yB'_y + A'_zB'_z = A_xB_x + A_yB_y + A_zB_z$$

yields

$$A'_x = a_1A_x + b_1A_y + c_1A_z.$$

Similarly, by taking for $O'X'Y'Z'$, in turn, the frames in which **B** has the presentations (0, 1, 0) and (0, 0, 1) respectively, we find

$$A'_y = a_2A_x + b_2A_y + c_2A_z,$$
$$A'_z = a_3A_x + b_3A_y + c_3A_z.$$

These equations show that the three numbers (A_x, A_y, A_z) present, in the $OXYZ$ frame, a vector **A**; for all frames $O'X'Y'Z'$ may be obtained by suitably varying the *arbitrary* vector **B**. The fact that the granted independence in value of the expression $A_xB_x + A_yB_y + A_zB_z$ of the reference frame, where (B_x, B_y, B_z) is a presentation of an arbitrary vector **B**, implies that (A_x, A_y, A_z) is a presentation of a vector **A**, will be found very useful in developing the theory of vectors.

d. The vector product of two vectors. If we have two vectors, **A** and **B**, one of which is not a mere scalar multiple of the other, we may consider the plane formed by their representative segments through any point M. This plane determines, by a line perpendicular to it, one of two directions. We shall agree to choose that one along which a right-handed screw would advance when turned in the sense that would be required to make the representative segment of **A** assume the direction of the representative segment of **B**, the rotation being through the smaller of the two angles between the two representative segments. In order to present this direction we may use an arbitrary frame in which **A** is presented by (A_x, A_y, A_z) and **B** by (B_x, B_y, B_z); and if **C** is any vector perpendicular to the plane including **A** and **B**, we have

$$(\mathbf{C}.\mathbf{A}) = C_xA_x + C_yA_y + C_zA_z = 0,$$
$$(\mathbf{C}.\mathbf{B}) = C_xB_x + C_yB_y + C_zB_z = 0;$$

whence

$$C_x = k(A_yB_z - A_zB_y), \quad C_y = k(A_zB_x - A_xB_z),$$
$$C_z = k(A_xB_y - A_yB_x),$$

where k is a factor of proportionality which might, conceivably, depend on the particular frame used to present **A**, **B**, and **C**. That it does not, follows from the fact that the magnitude of **C** is given by

$$\begin{aligned} C^2 &= k^2[(A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2] \\ &= k^2[(A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) \\ &\qquad - (A_xB_x + A_yB_y + A_zB_z)^2] \\ &= k^2A^2B^2(1 - \cos^2\theta) \\ &= k^2A^2B^2\sin^2\theta; \end{aligned}$$

whence $$C = \pm\, kAB\sin\theta.$$

To see which of the two signs should be taken, we choose a frame whose x-axis lies along the representative segment of **A**, and whose y-axis is in the plane of the two segments and directed so as to make an acute angle with the representative segment of **B**. Then **C** is, by definition, directed along the positive z-axis, and its presentation is $(0, 0, C)$, while the presentations of **A** and **B** are $(A, 0, 0)$ and $(B\cos\theta, B\sin\theta, 0)$ respectively. Upon substituting these in the expressions for C_x, C_y, and C_z above, we find $C_x = 0$, $C_y = 0$, $C_z = kAB\sin\theta$, and therefore

$$C = kAB\sin\theta,$$

so that k must be positive, since θ, being less than two right angles, has its sine positive. We fix the magnitude of **C** by making $k = 1$, so that **C** has, in the $OXYZ$ frame, the presentation

$$\begin{gathered} C_x = (A_yB_z - A_zB_y), \quad C_y = (A_zB_x - A_xB_z), \\ C_z = (A_xB_y - A_yB_x). \end{gathered} \tag{3.2}$$

This vector is known as the vector product of **A** and **B** and is denoted by the symbol [**A.B**]. It follows from (3.2) that

$$\begin{aligned} \mathbf{C} &= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \end{aligned}$$

If **A** and **B** are the unit vectors **i** and **j**, it follows from the definition that [**i.j**] = **k**; similarly, [**j.k**] = **i**, [**k.i**] = **j**. Now the presentations of **i**, **j**, and **k** in the $O'X'Y'Z'$ frame are given, by the equations (2.5), as (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3), respectively, so that on presenting the vector equation $\mathbf{i} = [\mathbf{j}.\mathbf{k}]$ in the $O'X'Y'Z'$ frame we have the relations

$$a_1 = b_2c_3 - b_3c_2, \quad a_2 = b_3c_1 - b_1c_3, \quad a_3 = b_1c_2 - b_2c_1 \tag{3.3}$$

among the direction cosines of (2.1). There are similar equations for (b_1, b_2, b_3) and (c_1, c_2, c_3). Upon interchanging the rôles of the $OXYZ$ and the $O'X'Y'Z'$ frames, we obtain the relations

$$a_1 = b_2c_3 - b_3c_2, \quad b_1 = c_2a_3 - c_3a_2, \quad c_1 = a_2b_3 - a_3b_2, \text{ etc.} \tag{3.4}$$

These relations among the nine direction cosines which give the relative orientation of two Cartesian frames $OXYZ$ and $O'X'Y'Z'$ may be used to verify the fact that the process which furnishes the set of three numbers $(A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)$ in each frame $OXYZ$ actually presents a vector. Thus it readily follows from (2.5) that

$$\begin{aligned} A'_yB'_z - A'_zB'_y = (b_2c_3 - b_3c_2)(A_yB_z - A_zB_y) \\ + (c_2a_3 - c_3a_2)(A_zB_x - A_xB_z) \\ + (a_2b_3 - a_3b_2)(A_xB_y - A_yB_x), \end{aligned}$$

and, by (3.4), this is equal to

$$a_1(A_yB_z - A_zB_y) + b_1(A_zB_x - A_xB_z) + c_1(A_xB_y - A_yB_x).$$

Proceeding similarly for $A'_zB'_x - A'_xB'_z$ and $A'_xB'_y - A'_yB'_x$, we see that the sets of three numbers furnished by our process satisfy the criterion given by the equations (2.5).

The fact that the magnitude of the vector product is $AB \sin \theta$ gives the useful expression

$$AB \sin \theta = [(A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2]^{\frac{1}{2}} \tag{3.5}$$

for $\sin \theta$. The magnitude $AB \sin \theta$ of the vector product is the area of the parallelogram whose coterminous edges are representative segments of **A** and **B**.

It readily follows from the expression (3.2), or directly from the definition of the vector product, that

$$[\mathbf{B}.\mathbf{A}] = - [\mathbf{A}.\mathbf{B}]. \tag{3.6}$$

Thus the vector product of two vectors does not obey the commutative law; for when the order of the factors in the product is changed, the direction of the product vector is also changed. On the other hand, the distributive law is obeyed; for the first component in the $OXYZ$ frame of $[\mathbf{A}.(\mathbf{B}+\mathbf{C})]$, being $A_y(B_z+C_z)-A_z(B_y+C_y)$, is equivalent to

$$(A_yB_z-A_zB_y)+(A_yC_z-A_zC_y),$$

and similarly for the other two components. Hence

$$[\mathbf{A}.(\mathbf{B}+\mathbf{C})]=[\mathbf{A}.\mathbf{B}]+[\mathbf{A}.\mathbf{C}]. \qquad (3.7)$$

More generally we have

$$[(\mathbf{A}+\mathbf{B}).(\mathbf{C}+\mathbf{D})]=[\mathbf{A}.\mathbf{C}]+[\mathbf{A}.\mathbf{D}]+[\mathbf{B}.\mathbf{C}]+[\mathbf{B}.\mathbf{D}]. \qquad (3.8)$$

The distributive law may be freely applied when developing vector products, but care must be taken to preserve the proper order of the factors in the various products.

When one of two vectors **A** and **B** is a scalar multiple of the other, the two representative segments through a point lie along the same straight line and do not determine a plane. The vector product of two such vectors is defined as the zero vector; this is in accordance with (3.2). Using the distributive law, we see that the vector product of two vectors is not altered if a scalar multiple of either of the two factors is added to the other factor. Thus

$$[\mathbf{A}.(\mathbf{B}+m\mathbf{A})]=[\mathbf{A}.\mathbf{B}]=[(\mathbf{A}+n\mathbf{B}).\mathbf{B}], \qquad (3.9)$$

where m and n are any scalar quantities. For this reason, vector division is not defined; for if we tried to interpret $\mathbf{A}\div\mathbf{B}=\mathbf{C}$, we should seek the vector **C** such that either $[\mathbf{B}.\mathbf{C}]=\mathbf{A}$ or $[\mathbf{C}.\mathbf{B}]=\mathbf{A}$. There would then be two kinds of quotients, which might be distinguished by some such terms as "right" and "left." Neither type of quotient would be unique; for if **C** is any vector satisfying $[\mathbf{B}.\mathbf{C}]=\mathbf{A}$, then $\mathbf{C}+m\mathbf{B}$, where m is any scalar, may be substituted for **C** without affecting the product.*

e. The scalar triple product. If we have three vectors, **A**, **B**, and **C**, and take the scalar product of **A** into the vector product of **B** and **C**, we obtain the scalar quantity

$$(\mathbf{A}.[\mathbf{B}.\mathbf{C}])=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \qquad (3.10)$$

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* The consideration of $\mathbf{A}\div\mathbf{B}$, not as a vector but as an operation carrying a representative segment of **B** into a coterminous representative segment of **A**, led Sir William Hamilton to the study of *quaternions*. The name "quaternion" implies that the operation requires for its specification four numbers: three to describe the rotation of the representative segment of **B** into the same straight line as that of **A**, and a fourth to give the ratio of the lengths of the two representative segments. (See § 24.)

The verification, by means of (2.5), that this determinant does not depend in value on the frame chosen to present the vectors follows at once from the law of multiplication of determinants and the fact that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1$, which is an immediate consequence of the equations (3.4) and (2.1). We have at once, on interchanging the position of the rows of the determinant of (3.10),

$$(\mathbf{A}.[\mathbf{B}.\mathbf{C}]) = ([\mathbf{A}.\mathbf{B}].\mathbf{C}) = (\mathbf{B}.[\mathbf{C}.\mathbf{A}]), \text{ and so on.}$$

There is no ambiguity, then, in writing $(\mathbf{A}.[\mathbf{B}.\mathbf{C}])$ in the form $(\mathbf{A}.\mathbf{B}.\mathbf{C})$. It is known as the scalar triple product of the three vectors, and we have

$$\begin{aligned}(\mathbf{A}.\mathbf{B}.\mathbf{C}) &= (\mathbf{B}.\mathbf{C}.\mathbf{A}) = (\mathbf{C}.\mathbf{A}.\mathbf{B}) = -(\mathbf{A}.\mathbf{C}.\mathbf{B}) \\ &= -(\mathbf{B}.\mathbf{A}.\mathbf{C}) = -(\mathbf{C}.\mathbf{B}.\mathbf{A}).\end{aligned} \qquad (3.11)$$

Furthermore, it follows from (3.9) that

$$(\mathbf{A}.\mathbf{B}.\mathbf{C}) = (\mathbf{A}.\mathbf{B}.\mathbf{C} + m\mathbf{A} + n\mathbf{B}), \qquad (3.12)$$

where m and n are arbitrary scalar quantities.

Since the magnitude of $[\mathbf{B}.\mathbf{C}]$ is the area of the parallelogram formed by coterminous representative segments of **B** and **C**, and since $[\mathbf{B}.\mathbf{C}]$ is directed along the perpendicular to the plane of these segments, we find, by (3.1), that $(\mathbf{A}.\mathbf{B}.\mathbf{C})$ is the product of the area of the base of the parallelepiped, whose coterminous edges are representative segments of **A**, **B**, and **C**, respectively, by its altitude. Thus $(\mathbf{A}.\mathbf{B}.\mathbf{C})$ is the volume of the parallelepiped. It will have a positive sign if **A** makes an acute angle with the direction of $[\mathbf{B}.\mathbf{C}]$, and a negative sign if **A** makes an obtuse angle with this direction. The necessary and sufficient condition that the three coterminous representative segments of **A**, **B**, and **C** should lie in one plane is that $(\mathbf{A}.\mathbf{B}.\mathbf{C})$ be zero. The three vectors will then be said, briefly, to be *coplanar*. The possibility that one of the vectors may be zero is here included, since the zero vector has an indeterminate direction.

f. The vector triple product. Let us consider the vector product of **A** and the vector product of **B** and **C**. The x-component of the presentation of this vector in the $OXYZ$ frame is, by (3.2),

$$A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z).$$

By adding and subtracting the term $A_xB_xC_x$ this may be written in the form

$$B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)$$

or
$$(\mathbf{A}.\mathbf{C})B_x - (\mathbf{A}.\mathbf{B})C_x.$$

We have similar expressions for the other two components, so that we may write

$$[\mathbf{A}.[\mathbf{B}.\mathbf{C}]] = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}. \qquad (3.13)$$

It follows at once that

$$[\mathbf{A}.[\mathbf{B}.\mathbf{C}]] + [\mathbf{B}.[\mathbf{C}.\mathbf{A}]] + [\mathbf{C}.[\mathbf{A}.\mathbf{B}]] = 0. \qquad (3.13\,a)$$

Remembering that a change in the order of the factors of a vector product changes the sign of the product, we may make a double change and write

$$[\mathbf{A}.[\mathbf{B}.\mathbf{C}]] = [[\mathbf{C}.\mathbf{B}].\mathbf{A}] = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}. \qquad (3.14)$$

It is easy to remember the result (3.14) if we observe that the vector triple product, being a vector which is perpendicular to $[\mathbf{B}.\mathbf{C}]$, must lie in the plane of **B** and **C**, so that it is a linear combination of **B** and **C**. The scalar multiples are the scalar products of the other two vectors, and the positive sign goes with the *middle* vector. It will be observed that the vector triple product does not obey the commutative law.

Applications. If **A**, **B**, **C**, **D** are any four vectors, we may apply (3.12) and (3.14) to $[[\mathbf{A}.\mathbf{B}].[\mathbf{C}.\mathbf{D}]]$, and, remembering that $([\mathbf{A}.\mathbf{B}].\mathbf{C})$ is the scalar triple product $(\mathbf{A}.\mathbf{B}.\mathbf{C})$, and so on, we have

$$\begin{aligned}[[\mathbf{A}.\mathbf{B}].[\mathbf{C}.\mathbf{D}]] &= (\mathbf{A}.\mathbf{B}.\mathbf{D})\mathbf{C} - (\mathbf{A}.\mathbf{B}.\mathbf{C})\mathbf{D} \\ &= (\mathbf{A}.\mathbf{C}.\mathbf{D})\mathbf{B} - (\mathbf{B}.\mathbf{C}.\mathbf{D})\mathbf{A}. \qquad (3.15)\end{aligned}$$

From (3.15) we derive the relation

$$(\mathbf{A}.\mathbf{B}.\mathbf{C})\mathbf{D} = (\mathbf{B}.\mathbf{C}.\mathbf{D})\mathbf{A} + (\mathbf{C}.\mathbf{A}.\mathbf{D})\mathbf{B} + (\mathbf{A}.\mathbf{B}.\mathbf{D})\mathbf{C}, \quad (3.16)$$

connecting four arbitrary vectors **A**, **B**, **C**, and **D**. If **A**, **B**, and **C** are not coplanar, so that $(\mathbf{A}.\mathbf{B}.\mathbf{C})$ is different from zero, we derive, from (3.16), the equation

$$\mathbf{D} = \frac{(\mathbf{B}.\mathbf{C}.\mathbf{D})}{(\mathbf{A}.\mathbf{B}.\mathbf{C})}\mathbf{A} + \frac{(\mathbf{C}.\mathbf{A}.\mathbf{D})}{(\mathbf{A}.\mathbf{B}.\mathbf{C})}\mathbf{B} + \frac{(\mathbf{A}.\mathbf{B}.\mathbf{D})}{(\mathbf{A}.\mathbf{B}.\mathbf{C})}\mathbf{C}, \qquad (3.17)$$

which expresses an arbitrary vector as a linear combination of the three non-coplanar vectors **A**, **B**, and **C**.

Upon adding together the three equations of the type (3.15),

$$[[\mathbf{A}.\mathbf{D}].[\mathbf{B}.\mathbf{C}]] = (\mathbf{A}.\mathbf{B}.\mathbf{C})\mathbf{D} - (\mathbf{B}.\mathbf{C}.\mathbf{D})\mathbf{A},$$
$$[[\mathbf{B}.\mathbf{D}].[\mathbf{C}.\mathbf{A}]] = (\mathbf{A}.\mathbf{B}.\mathbf{C})\mathbf{D} - (\mathbf{C}.\mathbf{A}.\mathbf{D})\mathbf{B},$$
$$[[\mathbf{C}.\mathbf{D}].[\mathbf{A}.\mathbf{B}]] = (\mathbf{C}.\mathbf{D}.\mathbf{B})\mathbf{A} - (\mathbf{C}.\mathbf{D}.\mathbf{A})\mathbf{B},$$

we obtain the equation

$$2(\mathbf{A}.\mathbf{B}.\mathbf{C})\mathbf{D} = [[\mathbf{A}.\mathbf{D}].[\mathbf{B}.\mathbf{C}]] + [[\mathbf{B}.\mathbf{D}].[\mathbf{C}.\mathbf{A}]] + [[\mathbf{C}.\mathbf{D}].[\mathbf{A}.\mathbf{B}]]. \qquad (3.18)$$

Let us now suppose that we have a process furnishing us in each frame three numbers (D_x, D_y, D_z); and, furthermore, let us suppose that although it is not known that these sets of three numbers present a vector, it is known that, if **A** is an arbitrary vector, $(A_yD_z - A_zD_y, A_zD_x - A_xD_z, A_xD_y - A_yD_x)$ present a vector. It follows from the equation (3.18) that the three numbers (D_x, D_y, D_z) actually present a vector. For the right-hand side of (3.18) is a vector, the first term in it being the vector product of the given vector whose presentation in the $OXYZ$ frame is $(A_yD_z - A_zD_y, A_zD_x - A_xD_z, A_xD_y - A_yD_x)$ into the vector $[\mathbf{B}.\mathbf{C}]$; and so for the other terms. Of course, since we do not know *ab initio* that (D_x, D_y, D_z) actually present a vector in the $OXYZ$ frame, (3.18) must be regarded merely as standing for the three separate equations

$$2(\mathbf{A}.\mathbf{B}.\mathbf{C})D_x = (A_zD_x - A_xD_z)(B_xC_y - B_yC_x) - (A_xD_y - A_yD_x)(B_zC_x - B_xC_z) + \cdots, \text{ etc.}$$

In other words, a necessary and sufficient condition for (D_x, D_y, D_z) to present a vector is that $(A_yD_z - A_zD_y, A_zD_x - A_xD_z, A_xD_y - A_yD_x)$ present a vector, where (A_x, A_y, A_z) is the presentation of an arbitrary vector **A**. The same result follows from a consideration of the invariance of the scalar triple product $(\mathbf{C}.\mathbf{A}.\mathbf{D})$ and the equality $(\mathbf{C}.[\mathbf{A}.\mathbf{D}]) = ([\mathbf{C}.\mathbf{A}].\mathbf{D})$.

Another theorem in vector algebra will be found very useful later on. Let us consider the scalar product of two vector products. It is a matter of elementary algebra to verify that

$$([\mathbf{A}.\mathbf{B}].[\mathbf{C}.\mathbf{D}]) = (\mathbf{A}.\mathbf{C})(\mathbf{B}.\mathbf{D}) - (\mathbf{A}.\mathbf{D})(\mathbf{B}.\mathbf{C}), \qquad (3.19)$$

where **A**, **B**, **C**, and **D** are any four vectors. If **B** and **D** coincide we have

$$([\mathbf{A}.\mathbf{B}].[\mathbf{C}.\mathbf{B}]) = (\mathbf{A}.\mathbf{C})(\mathbf{B}.\mathbf{B}) - (\mathbf{A}.\mathbf{B})(\mathbf{B}.\mathbf{C}). \qquad (3.20)$$

If **A**, **B**, and **C** are unit vectors, and if their representative segments are drawn with a common initial point O, the end points of these segments will form the vertices of a spherical triangle ABC on the unit sphere which has O as center. Indicating, as usual, the angles of the spherical triangle by A, B, and C, and the opposite sides by a, b, and c respectively, we know that the magnitude of

$[\mathbf{A}.\mathbf{B}] = \sin c$ and that the magnitude of $[\mathbf{C}.\mathbf{B}] = \sin a$, while the angle between these two vector products is B. Hence we have, from (3.20),

$$\sin c \sin a \cos B = \cos b - \cos c \cos a,$$

or

$$\cos b = \cos c \cos a + \sin c \sin a \cos B. \tag{3.21}$$

If, in (3.20), we replace each of the unit vectors **A**, **B**, and **C** by the unit vector having the direction of the vector product of the other two, the spherical triangle ABC is replaced by the polar triangle $A'B'C'$, where $A' = \pi - a$, $B' = \pi - b$, $C' = \pi - c$, $a' = \pi - A$, $b' = \pi - B$, $c' = \pi - C$; and we derive from (3.21) the second spherical-triangle formula

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b. \tag{3.22}$$

4. Vector analysis. *a. The gradient vector.* If with each point of a curve, a surface, or a region of space there is associated a definite vector **A**, we have what is known as a *vector field.* Analytically we have a vector field when, in any frame $OXYZ$, the components (A_x, A_y, A_z) of a vector **A** are functions of the coördinates (x, y, z) of a variable point. The vector field may be called continuous when (A_x, A_y, A_z) are continuous functions of (x, y, z), and differentiable to the first order when the nine partial derivatives $\frac{\partial A_x}{\partial x}$, $\frac{\partial A_x}{\partial y}$, etc. exist at all points (x, y, z) under consideration. Unless the contrary is expressly mentioned, we shall deal only with continuous vector fields differentiable to any required order. A curve having the property that at each of its points the associated vector **A** has the direction of the tangent to the curve will be called a *vector line* of the field. The vector lines of a vector field are found by integrating the equations

$$\frac{dx}{A_x} = \frac{dy}{A_y} = \frac{dz}{A_z}.$$

A simple and important example of a vector field is obtained as follows. Let $\phi(x, y, z)$ be any scalar point function, that is, a function of the coördinates (x, y, z) of a variable point whose value is independent of the frame in which the coördinates are measured. When any other frame is introduced, ϕ will, in general, change its form, but its value at any particular point will be unaltered. Denoting the new form by ϕ', we have the identical relation $\phi' = \phi$; and, upon differentiating in turn with respect to x', y', and z', we obtain, by the rule for composite differentiation and the equations (2.3), the results

$$\begin{aligned}\frac{\partial\phi'}{\partial x'} &= a_1\frac{\partial\phi}{\partial x} + b_1\frac{\partial\phi}{\partial y} + c_1\frac{\partial\phi}{\partial z},\\ \frac{\partial\phi'}{\partial y'} &= a_2\frac{\partial\phi}{\partial x} + b_2\frac{\partial\phi}{\partial y} + c_2\frac{\partial\phi}{\partial z},\\ \frac{\partial\phi'}{\partial z'} &= a_3\frac{\partial\phi}{\partial x} + b_3\frac{\partial\phi}{\partial y} + c_3\frac{\partial\phi}{\partial z}.\end{aligned} \tag{4.1}$$

If we compare these equations with (2.5), we see that at any point (x, y, z) the three numbers $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$ present a vector. This vector is known as the gradient of the scalar point function ϕ, and is written **grad** ϕ. The square of its magnitude at any point is a scalar which is usually denoted by $\Delta_1\phi$, and we have the relation

$$\Delta_1\phi = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2. \tag{4.2}$$

If we draw the level surfaces $\phi = \text{constant}$, we have $d\phi = 0$, or $\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$, where (dx, dy, dz) present in the tangent plane, at the point (x, y, z), an arbitrary vector to that member of the family of level surfaces which passes through the point. This shows that **grad** ϕ is, at any point, normal to the level surface $\phi = \text{constant}$ passing through this point. It is, moreover, directed toward that side of the surface $\phi = \text{constant}$ on which ϕ is increasing. If we have any direction s whose direction cosines are (l, m, n), say, through (x, y, z), the resolved part of the gradient of ϕ in this direction is $l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}$, and this is known as the *directional derivative* of ϕ in the direction s. It is denoted by $\frac{d\phi}{ds}$. The directional derivative is accordingly a maximum, at any point, when s is the normal n to the level surface $\phi = \text{constant}$ through that point, the normal being drawn in the direction toward which ϕ is increasing. The maximum value is

$$\frac{d\phi}{dn} = \text{magnitude of } \mathbf{grad}\ \phi = (\Delta_1\phi)^{\frac{1}{2}}.$$

If θ is the angle between the normal n and any other direction s, we have

$$\frac{d\phi}{ds}=\frac{d\phi}{dn}\cos\theta. \tag{4.3}$$

b. The divergence of a vector field. If we have a vector field **A**, the equations (2.5) and (4.1) show that the derivative $\frac{\partial A_x}{\partial x}$ transforms, when a change of frame is made, just as does the product A_xB_x of the x-components of two vectors **A** and **B**. Thus

$$\begin{aligned}\frac{\partial A'_x}{\partial x'}&=\frac{\partial}{\partial x'}(a_1A_x+b_1A_y+c_1A_z)\\&=\left(a_1\frac{\partial}{\partial x}+b_1\frac{\partial}{\partial y}+c_1\frac{\partial}{\partial z}\right)(a_1A_x+b_1A_y+c_1A_z)\\&=a_1^2\frac{\partial A_x}{\partial x}+b_1^2\frac{\partial A_y}{\partial y}+c_1^2\frac{\partial A_z}{\partial z}+b_1c_1\left(\frac{\partial A_z}{\partial y}+\frac{\partial A_y}{\partial z}\right)+\cdots,\end{aligned}$$

since the direction cosines (a_1, b_1, c_1) are independent of (x, y, z). Upon adding the two similar expressions for $\frac{\partial A'_y}{\partial y'}$ and $\frac{\partial A'_z}{\partial z'}$, we find, on using (2.2), that

$$\frac{\partial A'_x}{\partial x'}+\frac{\partial A'_y}{\partial y'}+\frac{\partial A'_z}{\partial z'}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}.$$

We have, then, associated with any vector field **A** a scalar point function $\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$, which is known as the *divergence* of **A** and is written div **A**. Thus

$$\operatorname{div}\mathbf{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}. \tag{4.4}$$

If in the scalar product $A_xB_x+A_yB_y+A_zB_z$ we replace A_x by $\frac{\partial}{\partial x}$, A_y by $\frac{\partial}{\partial y}$, and A_z by $\frac{\partial}{\partial z}$, the expression becomes

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z},$$

or div **B**. In other words, if we use ∇ as a symbolic vector operator whose components are $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$, we have

$$\operatorname{div}\mathbf{B}=(\nabla.\mathbf{B}).$$

It follows from (4.4) that if ψ is any scalar point function,

$$\operatorname{div}\psi\mathbf{A}=\psi\operatorname{div}\mathbf{A}+(\mathbf{A}.\operatorname{grad}\psi). \tag{4.5}$$

When the vector **A** is the gradient of a scalar point function ϕ, its divergence takes the form

$$\text{div}\,\mathbf{A} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

The scalar point function which is derived in this way from an arbitrary point function ϕ is known as the Laplacian (or second differential parameter) of ϕ and may be denoted by $\Delta_2\phi$. Thus

$$\Delta_2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{4.6}$$

The Laplacian is the result of an *invariant operator*. Thus in the frame $O'X'Y'Z'$ the operator has the form Δ'_2, and we have just proved that the equation $\Delta'_2\phi' = \Delta_2\phi$ is a consequence of the equation $\phi' = \phi$. Such invariant operators are known as *differential parameters*. It is for this reason that the Laplacian is often referred to as the second differential parameter of ϕ. The result of the invariant differential operator Δ_1 of (4.2) is known as the first differential parameter of the scalar point function ϕ. If we form the expressions $(\Delta_2 A_x,\ \Delta_2 A_y,\ \Delta_2 A_z)$, where $(A_x,\ A_y,\ A_z)$ are the components in the $OXYZ$ frame of a vector **A**, it is easy to show that they present a vector. The fact that Δ_2 is an invariant operator enables us to put $\Delta'_2 A'_x = \Delta_2 A'_x$, where Δ'_2 is the operator $\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$. We have, then, from the equation

$$A'_x = a_1 A_x + b_1 A_y + c_1 A_z,$$

the result

$$\Delta'_2 A'_x = a_1\Delta_2 A_x + b_1\Delta_2 A_y + c_1\Delta_2 A_z,$$

since the direction cosines $(a_1,\ b_1,\ c_1)$ are independent of $(x,\ y,\ z)$. This, and the two similar equations for $\Delta'_2 A'_y$ and $\Delta'_2 A'_z$, show that $(\Delta_2 A_x,\ \Delta_2 A_y,\ \Delta_2 A_z)$ present a vector; this vector is denoted by the symbol $\Delta_2\mathbf{A}$.

c. The curl of a vector field. We have obtained above the scalar quantity $\text{div}\,\mathbf{A} = (\nabla.\mathbf{A})$, where ∇ is a vector operator whose components are $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$; let us now form the symbolic vector product $[\nabla.\mathbf{A}]$. This gives a vector whose components are

$$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z},\ \ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x},\ \ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \tag{4.7}$$

That these are actually the components in the $OXYZ$ frame of a vector follows at once by a proof similar to the analytic verification of the fact that the three components of the vector product $[\mathbf{A}.\mathbf{B}]$ of two vectors present a vector (see § 3, *d*). The vector presented by the components given in (4.7) is written **curl A**. (Sometimes the notation **rot A** is used.) The definition of the vector **curl A** may be put concisely in the form

$$\mathbf{curl\ A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}.$$

It follows at once that the curl of any gradient vector is zero, and that the divergence of the curl of any vector is zero. Thus

$$\mathbf{curl\ grad}\ \phi = 0, \quad \text{div}\ \mathbf{curl\ A} = 0. \tag{4.8}$$

If we wish to find the curl of a vector field which is itself the curl of another vector field, we have, for the first component of the vector **curl curl A**, the expression

$$\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right).$$

By adding and subtracting the quantity $\dfrac{\partial^2 A_x}{\partial x^2}$, this may be written in the form

$$\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right),$$

so that we have the result

$$\mathbf{curl\ curl\ A} = \mathbf{grad}\ \text{div}\ \mathbf{A} - \Delta_2 \mathbf{A}. \tag{4.9}$$

It is also easy to see that if ϕ is any scalar point function,

$$\mathbf{curl}\ \phi\mathbf{A} = \phi\ \mathbf{curl\ A} + [\mathbf{grad}\ \phi.\mathbf{A}]. \tag{4.10}$$

5. The derivative of a vector. Let us suppose that we have a vector field **A** attached to the points of a curve, and let the parameter along this curve be denoted by τ. Then the components (A_x, A_y, A_z) of **A** in any frame $OXYZ$ are functions of (x, y, z) and, through them, functions of the ultimate independent variable τ. The values of this variable are quite independent of the frame, and so we may differentiate the equations (2.5) with respect to τ, the direction cosines (a_1, b_1, c_1) etc.

being constants in the differentiation. We thus obtain three equations of the type

$$\frac{dA'_x}{d\tau} = a_1\frac{dA_x}{d\tau} + b_1\frac{dA_y}{d\tau} + c_1\frac{dA_z}{d\tau}.$$

These equations show that $\left(\frac{dA_x}{d\tau}, \frac{dA_y}{d\tau}, \frac{dA_z}{d\tau}\right)$ are the components, in the frame $OXYZ$, of a vector which we may denote by $\frac{d\mathbf{A}}{d\tau}$. From this new vector we may derive the vector $\frac{d^2\mathbf{A}}{d\tau^2}$, whose components are $\left(\frac{d^2A_x}{d\tau^2}, \frac{d^2A_y}{d\tau^2}, \frac{d^2A_z}{d\tau^2}\right)$, and so on. These results hold for any vector (not necessarily a vector field) whose components are functions of a certain parameter τ the values of which are independent of the reference frame. The essential fact in the proof is that the direction cosines (a_1, b_1, c_1) etc., which fix the relative orientation of any two frames, must be independent of τ.

The following results are immediate consequences of the definition of the derivative of a vector. If m is any scalar,

$$\frac{d}{d\tau}(m\mathbf{A}) = m\frac{d}{d\tau}\mathbf{A} + \frac{dm}{d\tau}\mathbf{A}, \tag{5.1}$$

$$\frac{d}{d\tau}(\mathbf{A}.\mathbf{B}) = \left(\mathbf{A}\cdot\frac{d\mathbf{B}}{d\tau}\right) + \left(\frac{d\mathbf{A}}{d\tau}\cdot\mathbf{B}\right). \tag{5.2}$$

In particular, $\frac{d}{d\tau}(\mathbf{A}.\mathbf{A}) = 2\left(\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right)$. Thus, if $\mathbf{A}$ is a vector of constant magnitude, so that $(\mathbf{A}.\mathbf{A})$ is constant, $\left(\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right) = 0$, which shows that the vector $\frac{d\mathbf{A}}{d\tau}$ is perpendicular to $\mathbf{A}$. If, for example, we know that $\mathbf{A}$ is a unit vector, then it follows that

$$\frac{d\mathbf{A}}{d\tau} \text{ is perpendicular to } \mathbf{A}. \tag{5.3}$$

Furthermore, $$\frac{d}{d\tau}[\mathbf{A}.\mathbf{B}] = \left[\mathbf{A}\cdot\frac{d\mathbf{B}}{d\tau}\right] + \left[\frac{d\mathbf{A}}{d\tau}\cdot\mathbf{B}\right], \tag{5.4}$$

and, in particular,

$$\frac{d}{d\tau}\left[\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right] = \left[\mathbf{A}\cdot\frac{d^2\mathbf{A}}{d\tau^2}\right].$$

In general the ordinary rule for the differentiation of a product holds for the differentiation of vector products. Since, however, vector multiplication is not commutative, care must be taken to preserve the order of the factors of the vector products during the differentiation.

Let us now consider the special case where **A** is the vector **r** whose presentation in the frame $OXYZ$ is (x, y, z). The vector **r** is known as the position vector of the point M whose coördinates are (x, y, z), and it has the segment from O to M as a representative segment. Let M trace out a curve along which the parameter is τ. Then the vector $\frac{d}{d\tau}\mathbf{r}$ has, by definition, the direction of the tangent to the curve traced out by M, drawn in the direction in which τ is increasing. If s denotes the length of arc along the curve measured from some convenient point in the direction in which τ is increasing, the unit vector in the direction of the tangent is $\frac{d}{ds}\mathbf{r}$. This may be denoted by $\mathbf{t}_1$; so that we have

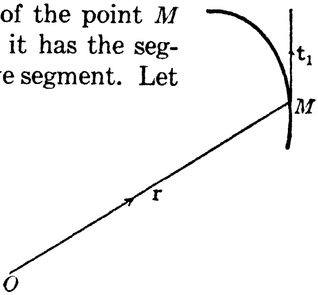

FIG. 3

$$\frac{d}{d\tau}\mathbf{r} = v\mathbf{t}_1,$$

where
$$v = \frac{ds}{d\tau} = \left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right]^{\frac{1}{2}}.$$

From this we find, by (5.1),

$$\frac{d^2\mathbf{r}}{d\tau^2} = v\frac{d\mathbf{t}_1}{d\tau} + \frac{dv}{d\tau}\mathbf{t}_1.$$

Since $\mathbf{t}_1$ is a unit vector, $\frac{d\mathbf{t}_1}{d\tau}$ is perpendicular to it; furthermore, $\frac{d\mathbf{t}_1}{d\tau}$ lies in the osculating plane of the curve at the point M, that is, the plane of two consecutive tangents at the point M. Its direction is, by definition, that of the principal normal to the curve. Denoting the unit vector along this principal normal by **n**, the equation

$$\frac{d\mathbf{t}_1}{ds} = \frac{\mathbf{n}}{\rho} \tag{5.5}$$

defines the positive quantity ρ, which is called the radius of curvature of the curve at the point M. Since, then, $\frac{d\mathbf{t}_1}{d\tau} = v\frac{\mathbf{n}}{\rho}$, we have the result

$$\frac{d^2\mathbf{r}}{d\tau^2} = \frac{v^2}{\rho}\mathbf{n} + \frac{dv}{d\tau}\mathbf{t}_1. \tag{5.6}$$

Thus the vector $\dfrac{d^2\mathbf{r}}{d\tau^2}$ lies entirely in the osculating plane of the curve and has components of magnitude $\dfrac{dv}{d\tau}$ and $\dfrac{v^2}{\rho}$ along the tangent and the principal normal respectively. When the curve is plane, the osculating plane is the plane of the curve, and the principal normal is the ordinary normal directed toward the center of curvature.

6. Orthogonal curvilinear coördinates. It is frequently convenient to use coördinates other than Cartesian to specify the positions of the various points of a material body. As familiar examples we have cylindrical coördinates and space polar coördinates. In cylindrical coördinates the three coördinates (ρ, ϕ, z) are, respectively, the distance ρ from a given line, the angle ϕ from a fixed half-plane through this line to the half-plane through the point (ρ, ϕ, z) and the line, and, finally, the distance z along the given line from a given point on it, known as the origin, to its intersection with a plane through (ρ, ϕ, z) perpendicular to it. The distance z is measured positively in the direction of advance of a right-handed screw which is turned in the direction in which ϕ is measured positively. The quantity ρ is essentially positive, ϕ may be taken to lie in the interval $0 \leqq \phi < 2\pi$, and z can take on any value, positive, negative, or zero. If we introduce a Cartesian frame with its origin at the origin of the cylindrical coördinates and with its z-axis coinciding with the z-axis of the cylindrical coördinates, while the x-axis is in the standard half-plane through the z-axis (that is, the half-plane from which ϕ is measured), we have the obvious relationship

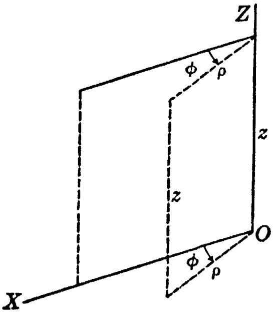

FIG. 4

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z. \tag{6.1}$$

Space polar coördinates differ from cylindrical coördinates merely in substituting for z and ρ the angle θ from the z-axis to the radius vector r from the origin to the point whose position is being specified, and the length of this radius vector. We have, therefore,

$$\rho = r \sin \theta, \quad z = r \cos \theta;$$

so that (6.1) yields

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \tag{6.2}$$

The radius r is essentially positive, θ lies in the interval $0 \leqq \theta \leqq \pi$, and ϕ, as before, lies in the interval $0 \leqq \phi < 2\pi$.

Through each point there pass, in either case, three coördinate surfaces, namely, the surfaces along which one of the curvilinear coördinates is constant, the other two being allowed to vary. Thus, in cylindrical coördinates the coördinate surfaces are, respectively, a circular cylinder with its axis along the z-axis, a half-plane through the z-axis, and a plane perpendicular to the z-axis. In space polar coördinates they are, respectively, a sphere with its center at the origin, a half-cone with its vertex at the origin, and a half-plane through the z-axis. These coördinate surfaces intersect, two by two, in curves which are known as coördinate curves, the intersection of the second and third coördinate surfaces being called the first coördinate curve, and so on. The parametric equations of the coördinate curves, with reference to the Cartesian frame, are given by the equations (6.1) or (6.2), in which the single parameter is the single varying coördinate. In cylindrical coördinates the coördinate curves (or lines) are, respectively, a half-line perpendicular to the z-axis, a circle with its center on the z-axis and lying in a plane perpendicular to the z-axis, and, finally, a straight line parallel to the z-axis. In space polar coördinates they are, respectively, a half-line through the origin, a semicircle with center at the origin and with its end points on the z-axis, and, finally, a circle with its center on the z-axis and lying in a plane perpendicular to the z-axis.

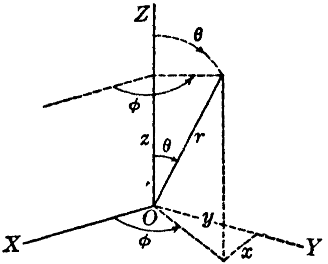

FIG. 5

The tangent lines to the three coördinate curves through any point, drawn in the direction in which the corresponding coördinate is increasing, furnish us with what are known as the three coördinate directions at that point. The direction cosines, relative to the Cartesian frame, of the coördinate directions follow readily from the equations (6.1) and (6.2). Thus, to get the direction cosines of the first coördinate line in cylindrical coördinates, we have merely to calculate $\left(\frac{\partial x}{\partial \rho}, \frac{\partial y}{\partial \rho}, \frac{\partial z}{\partial \rho}\right)$

and to divide each of these quantities by the positive square root of the sum of their squares. In this way we find the following tables of direction cosines for cylindrical and space polar coördinates respectively:

	x	y	z
ρ	$\cos\phi$	$\sin\phi$	0
ϕ	$-\sin\phi$	$\cos\phi$	0
z	0	0	1

(6.3)

	x	y	z
r	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
θ	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
ϕ	$-\sin\phi$	$\cos\phi$	0

(6.4)

It is seen at once that these direction cosines satisfy in each case the relations (2.1) or (2.2), so that the coördinate directions at any point form a set of three mutually perpendicular straight lines. This is expressed by the statement that cylindrical coördinates and space polar coördinates each constitute a system of orthogonal curvilinear coördinates. It is convenient to consider the Cartesian reference frame formed by the tangents to the three coördinate lines at any point. We have, then, attached to every point a Cartesian frame, and the directions of the axes of the frame vary from point to point. If we specify the directions of any one of these frames by means of the direction cosines of its axes relative to a fixed Cartesian frame $OXYZ$, we have the tables of direction cosines (6.3) and (6.4). The essential thing to notice is that the cosines of these tables are not, like the direction cosines of the equations (2.3), independent of (x, y, z).

We may now consider the general idea of curvilinear coördinates, of which cylindrical and space polar coördinates are but special, although important, instances. We shall have as coördinates three functions (α, β, γ) of (x, y, z), defined by three equations of the type

$$\alpha = \alpha(x, y, z), \quad \beta = \beta(x, y, z), \quad \gamma = \gamma(x, y, z), \qquad (6.5)$$

which we shall suppose solvable for (x, y, z); so that we shall have three equations of the type

$$x = x(\alpha, \beta, \gamma), \quad y = y(\alpha, \beta, \gamma), \quad z = z(\alpha, \beta, \gamma). \tag{6.6}$$

It will be seen that (6.1) and (6.2) are special instances of (6.6). The coördinate surfaces are obtained by setting, in turn, α constant, β constant, and γ constant in (6.5), and these coördinate surfaces furnish by their intersections the three coördinate lines through any point. These give, by means of their tangents, a rectangular Cartesian reference frame, since we shall so choose the coördinates that the coördinate surfaces are everywhere at right angles. We shall also so arrange matters that these Cartesian frames are all right-handed.* The table of direction cosines for the Cartesian frame attached to any point is

	x	y	z
α	$h_1 \dfrac{\partial x}{\partial \alpha}$	$h_1 \dfrac{\partial y}{\partial \alpha}$	$h_1 \dfrac{\partial z}{\partial \alpha}$
β	$h_2 \dfrac{\partial x}{\partial \beta}$	$h_2 \dfrac{\partial y}{\partial \beta}$	$h_2 \dfrac{\partial z}{\partial \beta}$
γ	$h_3 \dfrac{\partial x}{\partial \gamma}$	$h_3 \dfrac{\partial y}{\partial \gamma}$	$h_3 \dfrac{\partial z}{\partial \gamma}$

(6.7)

Here $$h_1 = \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2\right]^{-\frac{1}{2}} \text{etc.} \tag{6.8}$$

The condition that the curvilinear coördinates should be orthogonal is expressed by three equations of the type

$$\frac{\partial x}{\partial \beta}\frac{\partial x}{\partial \gamma} + \frac{\partial y}{\partial \beta}\frac{\partial y}{\partial \gamma} + \frac{\partial z}{\partial \beta}\frac{\partial z}{\partial \gamma} = 0. \tag{6.9}$$

The three coördinate directions at any point are, in addition to being tangent to the coördinate curves, normal to the coördinate surfaces. We obtain, then, from the equations (6.5), the following table of direction cosines, which, though different in form, are identical in value with those of (6.7):

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*In writing the equations (6.5) and (6.6) we make the assumption that the Jacobian determinant $\dfrac{\partial (x, y, z)}{\partial (\alpha, \beta, \gamma)}$ does not vanish for any point (α, β, γ) in which we are interested. Further, we assume that the partial derivatives $\dfrac{\partial x}{\partial \alpha}$ etc. are continuous, so that the Jacobian cannot change sign since it is continuous and is supposed not to vanish. In order to insure that the Cartesian frames attached to the various points are all right-handed, it is necessary merely to arrange that $\dfrac{\partial (x, y, z)}{\partial (\alpha, \beta, \gamma)}$ is positive at some one point (α, β, γ). It will then be positive at all points (α, β, γ), and all the frames will be right-handed.

	x	y	z
α	$k_1\frac{\partial\alpha}{\partial x}$	$k_1\frac{\partial\alpha}{\partial y}$	$k_1\frac{\partial\alpha}{\partial z}$
β	$k_2\frac{\partial\beta}{\partial x}$	$k_2\frac{\partial\beta}{\partial y}$	$k_2\frac{\partial\beta}{\partial z}$
γ	$k_3\frac{\partial\gamma}{\partial x}$	$k_3\frac{\partial\gamma}{\partial y}$	$k_3\frac{\partial\gamma}{\partial z}$

(6.10)

where
$$k_1 = \left[\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2\right]^{-\frac{1}{2}}\text{etc.} \qquad (6.11)$$

The identity of the tables (6.7) and (6.10) gives us nine equations of the type
$$h_1\frac{\partial x}{\partial\alpha} = k_1\frac{\partial\alpha}{\partial x}; \qquad (6.12)$$

and so we have, from (6.8), the equation
$$\frac{1}{h_1^2} = \left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2+\left(\frac{\partial z}{\partial\alpha}\right)^2$$
$$= \frac{k_1}{h_1}\left[\frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial\alpha}+\frac{\partial\alpha}{\partial y}\frac{\partial y}{\partial\alpha}+\frac{\partial\alpha}{\partial z}\frac{\partial z}{\partial\alpha}\right]. \qquad \text{[By (6.12)]}$$

The expression in brackets is, by the rule for composite differentiation, merely the partial derivative of α with respect to α, where β and γ are the other independent variables. Its value is accordingly unity, and we have the result

$$k_1 = \frac{1}{h_1},$$
and, similarly,
$$k_2 = \frac{1}{h_2}, \quad k_3 = \frac{1}{h_3}. \qquad (6.13)$$

If the curvilinear coördinates (α, β, γ) and the Cartesian coördinates (x, y, z) are each numbered 1, 2, and 3, respectively, we may indicate the general cosine of the table (6.7) by the symbol c_s^r, where r and s may take, independently, any one of the three values 1, 2, and 3. The upper label refers to the coördinate line, and the lower to the Cartesian axis. For example, c_3^2 is the cosine of the angle between the second coördinate curve and the third Cartesian axis, that is, between the β and z directions.

Denoting, for the moment, (α, β, γ) by $(\alpha_1, \alpha_2, \alpha_3)$ and (x, y, z) by (x_1, x_2, x_3), we have the general formula

$$c_s^r = h_r \frac{\partial x_s}{\partial \alpha_r} = \frac{1}{h_r} \frac{\partial \alpha_r}{\partial x_s}. \tag{6.14}$$

This formula is readily remembered if it is noticed that the h's and α's carry the same subscripts and occur in opposite places, one above and one below. (The suffix carried by the h's and the α's is the upper one on the c's.)

Let us now consider a vector **A** whose presentation in the Cartesian frame $OXYZ$ is (A_x, A_y, A_z), and whose presentation in the Cartesian frame attached to the point (α, β, γ) is $(A_\alpha, A_\beta, A_\gamma)$. We have then, from (2.5) and the definitions of the c_s^r, three equations of the type $A_\alpha = c_1^1 A_x + c_2^1 A_y + c_3^1 A_z$. If we are particularly interested in a point P_0, whose curvilinear coördinates are $(\alpha_0, \beta_0, \gamma_0)$, it is convenient to choose the frame $OXYZ$ so that its axes have the same directions as those of the frame attached to P_0. We have, then,

$$\begin{aligned} &(c_r^r)_0 = 1, \quad (r = 1, 2, 3) \\ \text{and} \qquad &(c_s^r)_0 = 0; \quad (r \neq s) \end{aligned} \tag{6.15}$$

so that $$(A_\alpha)_0 = A_x, \quad (A_\beta)_0 = A_y, \quad (A_\gamma)_0 = A_z. \tag{6.16}$$

From (6.14) and (6.15) we derive

$$\begin{aligned} &\left(\frac{\partial \alpha_r}{\partial x_r}\right)_0 = (h_r)_0, \quad (r = 1, 2, 3) \\ \text{and} \qquad &\left(\frac{\partial \alpha_r}{\partial x_s}\right)_0 = 0. \qquad (r \neq s) \end{aligned} \tag{6.17}$$

The subscript zero is used to indicate the values at the point P_0.

If $\mathbf{A} = \mathbf{grad}\,\phi$, where ϕ is a scalar point function, we have, by (6.16),

$$\begin{aligned} (A_\alpha)_0 &= \left(\frac{\partial \phi}{\partial x}\right)_0 = \left(\frac{\partial \phi}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial \phi}{\partial \beta}\frac{\partial \beta}{\partial x} + \frac{\partial \phi}{\partial \gamma}\frac{\partial \gamma}{\partial x}\right)_0 \\ &= \left(h_1 \frac{\partial \phi}{\partial \alpha}\right)_0. \qquad \text{[By (6.17)]} \end{aligned}$$

Since P_0 is any point whatsoever, we may now drop the subscript zero; and we see that the presentation, in the frame attached to the point (α, β, γ), of **grad** ϕ is

$$\left(h_1 \frac{\partial \phi}{\partial \alpha}, \quad h_2 \frac{\partial \phi}{\partial \beta}, \quad h_3 \frac{\partial \phi}{\partial \gamma}\right). \tag{6.18}$$

When we have a vector field and wish to find its divergence or curl, the fact that the direction cosines c_s^r are functions of (α, β, γ) is important. It is necessary to calculate the values at P_0 of the twenty-seven first derivatives of the nine cosines c_s^r with respect to x, y, and z. These may be denoted by $\left(\frac{\partial c_s^r}{\partial x_t}\right)_0$, where r, s, and t take, independently, any one of the values 1, 2, 3. Let us differentiate with respect to x_t the equations $c_r^1c_s^1 + c_r^2c_s^2 + c_r^3c_s^3 = 1$ or 0 (according as $s =$ or $\neq r$), which are a consequence of the orthogonality of the curvilinear coördinates; and let us put in, after differentiation, the values of the $(c_s^r)_0$ given in (6.15). We find

$$\left(\frac{\partial c_r^s}{\partial x_t}\right)_0 + \left(\frac{\partial c_s^r}{\partial x_t}\right)_0 = 0, \tag{6.19}$$

and in particular, when $r = s$, $\left(\frac{\partial c_r^r}{\partial x_t}\right)_0 = 0. \quad (6.20)$

Thus nine of the twenty-seven first derivatives of the c_s^r vanish at P_0, and the remaining eighteen are equal and opposite, in pairs. We have now merely to consider the case $r \neq s$. Since $\left(\frac{\partial \alpha_r}{\partial x_s}\right)_0 = 0$, from (6.17), the differentiation of $c_s^r = \frac{1}{h_r}\frac{\partial \alpha_r}{\partial x_s}$ with respect to x_t yields

$$\left(\frac{\partial c_s^r}{\partial x_t}\right)_0 = \left(\frac{1}{h_r}\frac{\partial^2 \alpha_r}{\partial x_s \partial x_t}\right)_0. \tag{6.21}$$

Hence, if s and t are each different from r, we have

$$\left(\frac{\partial c_s^r}{\partial x_t}\right)_0 = \left(\frac{\partial c_t^r}{\partial x_s}\right)_0. \tag{6.22}$$

Upon combining this result with (6.19), we can show that all the derivatives of the type $\frac{\partial c_1^2}{\partial x_3}$, where the labels are all different, vanish at P_0. Thus, for example,

$$\left(\frac{\partial c_1^2}{\partial x_3}\right)_0 = \left(\frac{\partial c_3^2}{\partial x_1}\right)_0 = -\left(\frac{\partial c_2^3}{\partial x_1}\right)_0$$

and $$\left(\frac{\partial c_1^2}{\partial x_3}\right)_0 = -\left(\frac{\partial c_2^1}{\partial x_3}\right)_0 = -\left(\frac{\partial c_3^1}{\partial x_2}\right)_0 = \left(\frac{\partial c_1^3}{\partial x_2}\right)_0 = \left(\frac{\partial c_2^3}{\partial x_1}\right)_0;$$

whence, on addition, we find

$$2\left(\frac{\partial c_1^2}{\partial x_3}\right)_0 = 0.$$

We may therefore write

$$\left(\frac{\partial c_r^s}{\partial x_t}\right)_0 = 0. \quad (r \neq s \neq t) \tag{6.23}$$

It remains to calculate the values of the six derivatives of the type $\left(\frac{\partial c_s{}^r}{\partial x_r}\right)_0$, where $r \neq s$. We have, from (6.21),

$$\left(\frac{\partial c_s{}^r}{\partial x_r}\right)_0 = \left(\frac{1}{h_r}\frac{\partial^2 \alpha_r}{\partial x_r \partial x_s}\right)_0. \quad (r \neq s) \tag{6.24}$$

It follows, from (6.11) and (6.13), that

$$h_r{}^2 = \left(\frac{\partial \alpha_r}{\partial x_1}\right)^2 + \left(\frac{\partial \alpha_r}{\partial x_2}\right)^2 + \left(\frac{\partial \alpha_r}{\partial x_3}\right)^2. \tag{6.25}$$

Upon differentiating this with respect to x_s and substituting from (6.17), we find

$$\left(\frac{\partial h_r}{\partial x_s}\right)_0 = \left(\frac{\partial^2 \alpha_r}{\partial x_r \partial x_s}\right)_0, \tag{6.26}$$

so that

$$\left(\frac{\partial c_s{}^r}{\partial x_r}\right)_0 = \left(\frac{1}{h_r}\frac{\partial h_r}{\partial x_s}\right)_0 = -\left(h_r \frac{\partial}{\partial x_s}\frac{1}{h_r}\right)_0. \tag{6.27}$$

On using (6.19) and (6.18) we find

$$\left(\frac{\partial c_r{}^s}{\partial x_r}\right)_0 = -\left(\frac{1}{h_r}\frac{\partial h_r}{\partial x_s}\right)_0 = \left(h_r \frac{\partial}{\partial x_s}\frac{1}{h_r}\right)_0 = \left(h_r h_s \frac{\partial}{\partial \alpha_s}\frac{1}{h_r}\right)_0. \tag{6.28}$$

Hence

$$\left(\frac{\partial c_s{}^r}{\partial x_r}\right)_0 = \left(\frac{h_s}{h_r}\frac{\partial h_r}{\partial \alpha_s}\right)_0 = -\left(h_r h_s \frac{\partial}{\partial \alpha_s}\frac{1}{h_r}\right)_0. \tag{6.29}$$

Let us now proceed to calculate the divergence of a vector field **A** which is presented, at each point (α, β, γ), by its components $(A_\alpha, A_\beta, A_\gamma)$ in the frame attached to (α, β, γ). We have the equations

$$A_x = c_1{}^1 A_\alpha + c_1{}^2 A_\beta + c_1{}^3 A_\gamma, \text{ etc.}$$

Consequently, by use of (6.15), (6.18), and (6.28),

$$\begin{aligned}\left(\frac{\partial A_x}{\partial x}\right)_0 &= \left(\frac{\partial A_\alpha}{\partial x}\right)_0 + \left(A_\beta \frac{\partial c_1{}^2}{\partial x} + A_\gamma \frac{\partial c_1{}^3}{\partial x}\right)_0 \\ &= \left(h_1 \frac{\partial A_\alpha}{\partial \alpha}\right)_0 + \left(A_\beta h_2 h_1 \frac{\partial}{\partial \beta}\frac{1}{h_1} + A_\gamma h_1 h_3 \frac{\partial}{\partial \gamma}\frac{1}{h_1}\right)_0.\end{aligned}$$

Since P_0 is an arbitrary point, we may drop the subscript zero and write

$$\frac{\partial A_x}{\partial x} = h_1\left(\frac{\partial A_\alpha}{\partial \alpha} + h_2 A_\beta \frac{\partial}{\partial \beta}\frac{1}{h_1} + h_3 A_\gamma \frac{\partial}{\partial \gamma}\frac{1}{h_1}\right), \tag{6.30}$$

where (x, y, z) are coördinates in a fixed Cartesian frame having the same orientation as the frame which is attached to the point (α, β, γ).

Upon adding the three equations of the type (6.30), we obtain

$$\operatorname{div} \mathbf{A} = h_1h_2h_3\left[\frac{\partial}{\partial\alpha}\left(\frac{A_\alpha}{h_2h_3}\right) + \frac{\partial}{\partial\beta}\left(\frac{A_\beta}{h_3h_1}\right) + \frac{\partial}{\partial\gamma}\left(\frac{A_\gamma}{h_1h_2}\right)\right]. \tag{6.31}$$

In particular, when $\mathbf{A} = \mathbf{grad}\, \phi$ we obtain, from (6.18) and (6.31),

$$\Delta_2\phi = h_1h_2h_3\left[\frac{\partial}{\partial\alpha}\left(\frac{h_1}{h_2h_3}\frac{\partial\phi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_2}{h_3h_1}\frac{\partial\phi}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h_3}{h_1h_2}\frac{\partial\phi}{\partial\gamma}\right)\right]. \tag{6.32}$$

The presentation of curl **A** in the frame attached to the point (α, β, γ) readily follows. From

$$A_z = c_3{}^1A_\alpha + c_3{}^2A_\beta + c_3{}^3A_\gamma$$

and the relations (6.20) and (6.23), we have

$$\left(\frac{\partial A_z}{\partial y}\right)_0 = \left(\frac{\partial A_\gamma}{\partial y}\right)_0 + \left(A_\beta\frac{\partial c_3{}^2}{\partial y}\right)_0.$$

This, by (6.18) and (6.29), is equal to

$$\left(h_2\frac{\partial A_\gamma}{\partial\beta}\right)_0 - \left(h_2h_3A_\beta\frac{\partial}{\partial\gamma}\frac{1}{h_2}\right)_0.$$

Upon combining this with the similar result for $\left(\frac{\partial A_y}{\partial z}\right)_0$, we have

$$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)_0 = \left\{h_2h_3\left[\frac{\partial}{\partial\beta}\left(\frac{A_\gamma}{h_3}\right) - \frac{\partial}{\partial\gamma}\left(\frac{A_\beta}{h_2}\right)\right]\right\}_0;$$

and, since P_0 is an arbitrary point, we may drop the subscript as before and write the presentation of curl **A** in the frame attached to (α, β, γ) as

$$h_2h_3\left[\frac{\partial}{\partial\beta}\left(\frac{A_\gamma}{h_3}\right) - \frac{\partial}{\partial\gamma}\left(\frac{A_\beta}{h_2}\right)\right],\quad h_3h_1\left[\frac{\partial}{\partial\gamma}\left(\frac{A_\alpha}{h_1}\right) - \frac{\partial}{\partial\alpha}\left(\frac{A_\gamma}{h_3}\right)\right],$$
$$h_1h_2\left[\frac{\partial}{\partial\alpha}\left(\frac{A_\beta}{h_2}\right) - \frac{\partial}{\partial\beta}\left(\frac{A_\alpha}{h_1}\right)\right]. \tag{6.33}$$

7. Special cases. The simplest way to remember the expressions for h_1, h_2, and h_3 is to notice that the square of the element of arc ds along any curve is given by the formula

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \frac{(d\alpha)^2}{h_1{}^2} + \frac{(d\beta)^2}{h_2{}^2} + \frac{(d\gamma)^2}{h_3{}^2}.$$

We have, then, the following results:

a. Cylindrical coördinates.

$$h_1 = 1, \quad h_2 = \frac{1}{\rho}, \quad h_3 = 1.$$

$$\text{grad } V = \left(\frac{\partial V}{\partial \rho}, \ \frac{1}{\rho}\frac{\partial V}{\partial \phi}, \ \frac{\partial V}{\partial z}\right). \tag{7.1}$$

$$\begin{aligned} \Delta_2 V &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial V}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial V}{\partial z}\right)\right] \\ &= \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}. \end{aligned} \tag{7.2}$$

$$\begin{aligned} \text{curl } \mathbf{A} = \frac{1}{\rho}\left[\frac{\partial A_z}{\partial \rho} - \frac{\partial}{\partial z}(\rho A_\phi)\right], \quad \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right], \\ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial}{\partial \phi}A_\rho\right]. \end{aligned} \tag{7.3}$$

b. Space polar coördinates.

$$h_1 = 1, \quad h_2 = \frac{1}{r}, \quad h_3 = \frac{1}{r \sin\theta}.$$

$$\text{grad } V = \left(\frac{\partial V}{\partial r}, \ \frac{1}{r}\frac{\partial V}{\partial \theta}, \ \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\right). \tag{7.4}$$

$$\begin{aligned} \Delta_2 V = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right)\right. \\ \left. + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\right)\right] \\ = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}. \end{aligned} \tag{7.5}$$

$$\begin{aligned} \text{curl } \mathbf{A} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial \theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial \phi}(rA_\theta)\right], \\ \frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(r\sin\theta A_\phi)\right], \quad \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right] \\ = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right], \\ \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi), \quad \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]. \end{aligned} \tag{7.6}$$

8. Tensors of the second rank. We have seen in § 3, *c*, that if **B** is an arbitrary vector, and if we are furnished in each frame $OXYZ$ with a set of three numbers such that the linear form

$$A_xB_x + A_yB_y + A_zB_z \tag{8.1}$$

in the components of the vector **B** has a value independent of the reference frame, then the set of three numbers (A_x, A_y, A_z) presents a vector. This point of view suggests an extension of the vector idea. Thus let us suppose that we have a process which furnishes us, for each reference frame $OXYZ$, with a set of nine quantities that we may conveniently arrange in a square as follows:

$$\begin{matrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{matrix} \tag{8.2}$$

If these nine quantities are such that the bilinear form

$$\begin{aligned} A_{xx}B_xC_x + A_{xy}B_xC_y + A_{xz}B_xC_z + A_{yx}B_yC_x + A_{yy}B_yC_y \\ + A_{yz}B_yC_z + A_{zx}B_zC_x + A_{zy}B_zC_y + A_{zz}B_zC_z \end{aligned} \tag{8.3}$$

in the components (B_x, B_y, B_z) and (C_x, C_y, C_z) of two arbitrary vectors **B** and **C** has a value independent of the reference frame, then the nine quantities (8.2) are said to present, in the frame $OXYZ$, a tensor of the second rank, of which the nine quantities themselves are said to be the components in that frame. This tensor we shall denote by the symbol $\mathfrak{A}$. The connection between the presentations in two different frames of the same tensor $\mathfrak{A}$ follows at once from the postulated invariance of the form (8.3) and from (2.5). Thus we may take, as special instances of the arbitrary vectors **B** and **C**, the vectors whose presentations in the $O'X'Y'Z'$ frame are each (1, 0, 0). Then, from (2.5), we have $B_x = C_x = a_1$, $B_y = C_y = b_1$, $B_z = C_z = c_1$. We find

$$\begin{aligned} A'_{xx} = a_1{}^2A_{xx} + a_1b_1(A_{xy} + A_{yx}) + a_1c_1(A_{xz} + A_{zx}) \\ + b_1{}^2A_{yy} + b_1c_1(A_{yz} + A_{zy}) + c_1{}^2A_{zz}, \end{aligned} \tag{8.4}$$

with similar formulas for the other components.

If **F** and **G** are any two vectors, the nine quantities

$$\begin{matrix} F_xG_x & F_xG_y & F_xG_z \\ F_yG_x & F_yG_y & F_yG_z \\ F_zG_x & F_zG_y & F_zG_z \end{matrix} \tag{8.5}$$

present, in the frame $OXYZ$, a tensor of the second rank, which is called the tensor product of the two vectors; for the bilinear form (8.3) becomes, on putting $A_{xx} = F_xG_x$ etc., equivalent to **(F.B)(G.C)**, and has, therefore, a value independent of the reference frame. The formulas (8.4) may, accordingly, be conveniently remembered by the statement that the components of a tensor of the second rank transform, on a change of reference frame, in exactly the same way as do the products of the components of two vectors.

It follows at once from the definition of a tensor of the second rank, by means of the invariance under change of reference frame of the bilinear form (8.3), that if $\mathfrak{A}$ and $\mathfrak{B}$ are two tensors of the second rank, the set of nine quantities

$$\begin{array}{ccc} A_{xx} + B_{xx} & A_{xy} + B_{xy} & A_{xz} + B_{xz} \\ A_{yx} + B_{yx} & A_{yy} + B_{yy} & A_{yz} + B_{yz} \\ A_{zx} + B_{zx} & A_{zy} + B_{zy} & A_{zz} + B_{zz} \end{array} \tag{8.6}$$

present, in the frame $OXYZ$, a tensor of the second rank, which we shall denote by $\mathfrak{A} + \mathfrak{B}$ and call the sum of $\mathfrak{A}$ and $\mathfrak{B}$. It is seen at once that the addition of tensors is commutative; that is, $\mathfrak{A} + \mathfrak{B} = \mathfrak{B} + \mathfrak{A}$. Similarly, if $\mathfrak{A}$ is any tensor, the set of nine quantities obtained by multiplying the components of $\mathfrak{A}$ by the same scalar quantity m present a tensor which will be denoted by $m\mathfrak{A}$. Multiplication of tensors by a scalar is seen to obey the distributive law; that is, $m(\mathfrak{A} + \mathfrak{B}) = m\mathfrak{A} + m\mathfrak{B}$. These definitions lead to the idea of the zero tensor, all of whose components in any frame are zero.

If an interchange of the rôles of the vectors **B** and **C** in the formation of the bilinear form (8.3) leaves its value unaltered, the tensor $\mathfrak{A}$ is said to be symmetric. The necessary and sufficient conditions for symmetry are easily seen to be

$$A_{yz} = A_{zy}, \quad A_{zx} = A_{xz}, \quad A_{xy} = A_{yx}. \tag{8.7}$$

If the equations (8.7) are satisfied in any one frame, they are satisfied in all frames, as may easily be verified by means of (8.4). If, on the other hand, an interchange of the rôles of **B** and **C** changes the sign of the bilinear form (8.3), but does not change its numeric value, the tensor $\mathfrak{A}$ is said to be alternating. The necessary and sufficient conditions for $\mathfrak{A}$ to be alternating are seen to be

$$\begin{array}{lll} A_{xx} = 0, & A_{yy} = 0, & A_{zz} = 0, \\ -A_{yz} = A_{zy}, & -A_{zx} = A_{xz}, & -A_{xy} = A_{yx}. \end{array} \tag{8.8}$$

As before, these equations are satisfied in all reference frames if they are satisfied in any particular frame. If we call the tensor derived from $\mathfrak{A}$ by an interchange of the vectors **B** and **C** in the bilinear form (8.3) the conjugate of $\mathfrak{A}$, we may say that a tensor is symmetric when it is equal to its conjugate, and alternating when it is the negative of its conjugate. Denoting the conjugate of $\mathfrak{A}$ by $\mathfrak{A}'$, we see from the identity

$$\mathfrak{A} = \tfrac{1}{2}[\mathfrak{A} + \mathfrak{A}'] + \tfrac{1}{2}[\mathfrak{A} - \mathfrak{A}']$$

that any tensor may be represented as the sum of two tensors one of which is symmetric and the other alternating.

9. Symmetric tensors. Let us suppose that the vectors **B** and **C** used in constructing the bilinear form (8.3) coincide, and let us draw that representative segment of either **B** or **C** whose initial point is at the origin of the $OXYZ$ frame. If the tensor $\mathfrak{A}$ is assumed symmetric, the end point (x, y, z) of the representative segment of the varying vector **B** traces out, for a given value (say, unity) of the scalar expression (8.3), the quadric surface

$$A_{xx}x^2 + A_{yy}y^2 + A_{zz}z^2 + 2\,A_{yz}yz + 2\,A_{zx}zx + 2\,A_{xy}xy = 1. \quad (9.1)$$

The surface may be regarded as a geometric representation of the symmetric tensor $\mathfrak{A}$. Conversely, any central quadric surface furnishes us with a unique symmetric tensor which it represents. In a particular frame, that formed by the principal axes of the quadric surface, the equation of the quadric takes the simple form

$$a_1X^2 + a_2Y^2 + a_3Z^2 = 1.$$

Consequently there is a frame in which the components of any symmetric tensor reduce to

$$\begin{matrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{matrix}$$

The axes of this frame are called the principal axes of the symmetric tensor. To determine these axes, we have to determine those directions of the radius vector of the general central quadric (9.1) for which $x^2 + y^2 + z^2$ is a maximum, a minimum, or a minimax; that is, we have to make $x^2 + y^2 + z^2$ stationary, where (x, y, z) are subject to the equation (9.1). For the tangent planes at the ends of the axes of the quadric surface (9.1) are perpendicular to the radii vectors at their respective points of contact, and consequently the lengths of these radii must be stationary at these points of contact. We have, then, connecting the differentials dx, dy, and dz, the two equations

$$x\,dx + y\,dy + z\,dz = 0,$$
$$(A_{xx}x + A_{xy}y + A_{xz}z)dx + (A_{yx}x + A_{yy}y + A_{yz}z)dy + (A_{zx}x + A_{zy}y + A_{zz}z)dz = 0.$$

We must have, therefore, for certain points on the quadric surface (9.1), the equations

$$\begin{aligned} A_{xx}x + A_{xy}y + A_{xz}z &= \lambda x, \\ A_{yx}x + A_{yy}y + A_{yz}z &= \lambda y, \\ A_{zx}x + A_{zy}y + A_{zz}z &= \lambda z, \end{aligned} \tag{9.2}$$

where λ is a factor of proportionality which will, in general, vary with (x, y, z). We see, from these three homogeneous equations for (x, y, z), that λ must satisfy the cubic equation

$$\begin{vmatrix} A_{xx} - \lambda & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} - \lambda & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} - \lambda \end{vmatrix} = 0. \tag{9.3}$$

The geometric significance of λ follows upon multiplication of the equations (9.2) by (x, y, z) respectively, and adding the products. We find $\lambda r^2 = 1$, where r is the length of the stationary radius vector. Corresponding to each root of the cubic (9.3) we have one of the principal axes; and, since the lengths of these axes do not depend on the reference frame, we see that the ratios of the coefficients of the λ cubic must be scalar quantities. For the cubic (9.3) must be identical with $(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0$, where $\lambda_1 = \dfrac{1}{r_1{}^2}$ etc., and where r_1, r_2, r_3 are the lengths of the semiaxes of the quadric surface (9.1). In this way we arrive at the three scalar quantities

$$S_1 = A_{xx} + A_{yy} + A_{zz}, \tag{9.4}$$
$$S_2 = A_{yy}A_{zz} + A_{zz}A_{xx} + A_{xx}A_{yy} - A_{yz}{}^2 - A_{zx}{}^2 - A_{xy}{}^2, \tag{9.5}$$
$$S_3 = \begin{vmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{vmatrix}, \tag{9.6}$$

formed from the components of any symmetric tensor $\mathfrak{A}$. These are known, respectively, as the first, second, and third scalar invariants of the symmetric tensor $\mathfrak{A}$.

The axes of a symmetric tensor are arbitrary when the representative quadric surface (9.1) is a sphere. In this case the tensor is a scalar multiple of the tensor

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \tag{9.7}$$

This tensor is remarkable for the fact that its presentation in all frames is the same. It arises naturally from (8.3) and from the known invariance of the scalar product

$$(\mathbf{B}.\mathbf{C}) = B_xC_x + B_yC_y + B_zC_z$$

of any two vectors.

10. Alternating tensors. When the tensor $\mathfrak{A}$ is alternating, the bilinear form (8.3) reduces to

$$\begin{aligned} A_{yz}(B_yC_z - B_zC_y) &+ A_{zx}(B_zC_x - B_xC_z) \\ &+ A_{xy}(B_xC_y - B_yC_x). \end{aligned} \tag{10.1}$$

Now **B** and **C** are arbitrary vectors, so that [**B**.**C**] is an arbitrary vector; therefore, the invariance of (10.1) tells us that the three quantities (A_{yz}, A_{zx}, A_{xy}) present a vector in the frame $OXYZ$ (see § 3, *c*). Conversely, we may construct from any vector **A** the alternating tensor

$$\begin{matrix} 0 & A_z & -A_y \\ -A_z & 0 & A_x \\ A_y & -A_x & 0 \end{matrix}$$

As an illustration of this consider the tensor product

$$\begin{matrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{matrix}$$

of two vectors **A** and **B** and its conjugate

$$\begin{matrix} A_xB_x & A_yB_x & A_zB_x \\ A_xB_y & A_yB_y & A_zB_y \\ A_xB_z & A_yB_z & A_zB_z \end{matrix}$$

The difference of these two tensors is the alternating tensor

$$\begin{matrix} 0 & A_xB_y - A_yB_x & A_xB_z - A_zB_x \\ A_yB_x - A_xB_y & 0 & A_yB_z - A_zB_y \\ A_zB_x - A_xB_z & A_zB_y - A_yB_z & 0 \end{matrix}$$

From this we derive, as above, the vector product [**A**.**B**] of two arbitrary vectors.

11. The inner product of a tensor and a vector. Collecting the terms of the scalar bilinear form (8.3) as follows,

$$(A_{xx}C_x + A_{xy}C_y + A_{xz}C_z)B_x + (A_{yx}C_x + A_{yy}C_y + A_{yz}C_z)B_y \\ + (A_{zx}C_x + A_{zy}C_y + A_{zz}C_z)B_z,$$

we see, since this is an invariant or scalar quantity for an arbitrary vector **B**, that the set of three quantities

$$(A_{xx}C_x + A_{xy}C_y + A_{xz}C_z, \quad A_{yx}C_x + A_{yy}C_y + A_{yz}C_z, \\ A_{zx}C_x + A_{zy}C_y + A_{zz}C_z)$$

presents, in the frame $OXYZ$, a vector (see § 3, *c*). In other words, if we are given a tensor $\mathfrak{A}$ and a vector **C**, the three quantities

$$\begin{aligned} A_{xx}C_x + A_{xy}C_y + A_{xz}C_z &= D_x, \\ A_{yx}C_x + A_{yy}C_y + A_{yz}C_z &= D_y, \\ A_{zx}C_x + A_{zy}C_y + A_{zz}C_z &= D_z \end{aligned} \tag{11.1}$$

are the components, in the frame $OXYZ$, of a vector **D**, called the inner product of the tensor $\mathfrak{A}$ and the vector **C**. The tensor $\mathfrak{A}$, when used in this way to associate with every vector **C** a vector **D**, is said to constitute a linear vector function.

If $\mathfrak{A}$ is symmetric and if we present it in the frame formed by its principal axes so that its components are

$$\begin{matrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{matrix}$$

our expressions (11.1) reduce to

$$D_x = a_1C_x, \quad D_y = a_2C_y, \quad D_z = a_3C_z. \tag{11.2}$$

The quadric surface which represents $\mathfrak{A}$ has the equation

$$a_1x^2 + a_2y^2 + a_3z^2 = 1. \tag{11.3}$$

Let (l, m, n) present the unit vector along **C** so that $C_x = Cl$, $C_y = Cm$, $C_z = Cn$. Then that representative segment of **C** whose initial point is at the center O of the quadric (11.3) meets this quadric in the point $x = rl$, $y = rm$, $z = rn$, where r is the length of the radius vector of the quadric in the direction of the segment. The tangent plane at this point has the equation

$$(x - rl)a_1l + (y - rm)a_2m + (z - rn)a_3n = 0, \tag{11.4}$$

and the origin O is on the negative side of this tangent plane at a distance $\frac{C}{rD}$. We see, then, from (11.2), that **D** is directed along the perpendicular from O to the tangent plane (11.4), its magnitude being given by the equation

$$D = \frac{C}{rp}, \tag{11.5}$$

where p is the length of the perpendicular. If we take $C = r$, we have $D = \frac{1}{p}$.

12. The analysis of tensor fields. When the components A_{xx} etc. of a tensor $\mathfrak{A}$ are functions of the coördinates (x, y, z) of a variable point, we have what is known as a tensor field. Thus, from two vector fields **A** and **B** we may derive the tensor field

$$\begin{matrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{matrix} \qquad \text{[See § 8]}$$

Now the equations (4.1) tell us that the quantity $\frac{\partial A_x}{\partial x}$ is transformed under a change of reference frame exactly like the expression A_xB_x. This leads us to the tensor

$$\begin{matrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_x}{\partial y} & \frac{\partial A_x}{\partial z} \\ \frac{\partial A_y}{\partial x} & \frac{\partial A_y}{\partial y} & \frac{\partial A_y}{\partial z} \\ \frac{\partial A_z}{\partial x} & \frac{\partial A_z}{\partial y} & \frac{\partial A_z}{\partial z} \end{matrix} \tag{12.1}$$

which is derived from a single vector field **A**. That (12.1) presents a tensor follows from § 5 and (8.3). Thus, upon differentiating **A** along any curve on which the parameter is τ, we obtain the vector

$$\frac{d\mathbf{A}}{d\tau} = \left(\frac{dA_x}{d\tau}, \quad \frac{dA_y}{d\tau}, \quad \frac{dA_z}{d\tau}\right).$$

The scalar product of this and an arbitrary vector **C** is the same for all reference frames; and on writing out

$$\frac{dA_x}{d\tau} = \frac{\partial A_x}{\partial x}\frac{dx}{d\tau} + \frac{\partial A_x}{\partial y}\frac{dy}{d\tau} + \frac{\partial A_x}{\partial z}\frac{dz}{d\tau},$$

we obtain for this scalar product an expression of the type (8.3), where $\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$ is the arbitrary vector **B** (arbitrary because the particular curve along which (x, y, z) varies is at our disposal), and where the coefficients of the bilinear form are given by (12.1). The first scalar invariant of the tensor (12.1) is div **A**, and the difference between (12.1) and its conjugate gives an alternating tensor which leads at once to the vector curl **A** (see § 10).

Similarly, the fact that the expressions (11.1) present a vector (**C** being an arbitrary vector), taken in conjunction with the equations (4.1), tells us that the three quantities

$$\begin{gathered}\frac{\partial A_{xx}}{\partial x}+\frac{\partial A_{xy}}{\partial y}+\frac{\partial A_{xz}}{\partial z},\\ \frac{\partial A_{yx}}{\partial x}+\frac{\partial A_{yy}}{\partial y}+\frac{\partial A_{yz}}{\partial z},\\ \frac{\partial A_{zx}}{\partial x}+\frac{\partial A_{zy}}{\partial y}+\frac{\partial A_{zz}}{\partial z}\end{gathered} \tag{12.2}$$

present a vector field. In this way we are able to derive from any tensor field $\mathfrak{A}$ a vector field. As an example, the vector field thus derived from the tensor (12.1) is $\Delta_2\mathbf{A}$. The significance of the vector field (12.2) will be seen after reading the next section on integrals.

13. Line, surface, and volume integrals. *a. Line integrals.* When the coördinates (x, y, z) of a variable point M are functions of a single independent variable τ, M traces out a curve C as τ varies. On differentiation with respect to τ the vector $\mathbf{r} = (x, y, z)$ yields the vector $\frac{d\mathbf{r}}{d\tau} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$. This vector is directed along the tangent to the curve at the point M; and its magnitude is $\frac{ds}{d\tau}$, where ds is the element of arc measured along the curve. Let us now suppose that we have a vector field **A** with given values at all points of the curve C; that is, (A_x, A_y, A_z) are known functions of the coördinates (x, y, z) of the variable point M on C. At any point M on C we may form the scalar product $\left(\mathbf{A}\cdot\frac{d\mathbf{r}}{d\tau}\right) = A_x\frac{dx}{d\tau} + A_y\frac{dy}{d\tau} + A_z\frac{dz}{d\tau}$; and on substituting for (x, y, z) in (A_x, A_y, A_z) their values in terms of τ, this becomes a function of the single independent variable τ. We may integrate it with respect to τ between any two values (say, τ_0 and τ_1) assumed by that variable at points $M_0 = (x_0, y_0, z_0)$ and $M_1 = (x_1, y_1, z_1)$ on the curve C. This definite integral is known as the line integral along C, between the points M_0 and M_1, of the vector **A**. It is quite possible that these two points will coincide even when τ_0 and τ_1 are different. In this case we have what is known as a line integral around a closed curve. As may readily be seen, the line integral

$$\int_{\tau_0}^{\tau_1}\left(A_x\frac{dx}{d\tau}+A_y\frac{dy}{d\tau}+A_z\frac{dz}{d\tau}\right)d\tau$$

has a value independent of the choice of the parameter τ, and so it is written in the form

$$\int_{M_0}^{M_1}(A_x\,dx+A_y\,dy+A_z\,dz), \tag{13.1}$$

in which no indication of the parameter to be used in the evaluation of the integral is given. The quantities (A_x, A_y, A_z) are known as the coefficients of the line integral, and the essential thing to notice is that the coefficients must present a vector if the value of the integrand of the line integral is to be independent of the reference frame in use. When the integral is to be taken over a closed curve, this is usually indicated by the symbol

$$\oint(A_x\,dx+A_y\,dy+A_z\,dz). \tag{13.2}$$

An important special case arises when the coefficients (A_x, A_y, A_z) of the line integral are the components, in the frame $OXYZ$, of a gradient vector. Then $A_x=\frac{\partial\phi}{\partial x}$, $A_y=\frac{\partial\phi}{\partial y}$, $A_z=\frac{\partial\phi}{\partial z}$, where ϕ is a scalar point function; and our line integral becomes

$$\int_{M_0}^{M_1}\left(\frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz\right)$$
$$=\int_{M_0}^{M_1}d\phi=\phi(x_1, y_1, z_1)-\phi(x_0, y_0, z_0).$$

It has, therefore, a value which depends merely on the end points of the curve along which the line integral is taken, and not on the form of the curve connecting these end points. The line integral is said in this case to be the integral of an exact differential.

b. Surface integrals. When the coördinates (x, y, z) of a variable point M are functions of two independent variables u and v, M traces out a surface S as u and v vary. Through each point of S there pass two curves which lie entirely on S and which are obtained by putting v and u, in turn, constant in the expressions for (x, y, z) as functions of u and v. These curves are called the parametric curves of the surface. The vector $\mathbf{r} = (x, y, z)$ yields, on partial differentiation with respect to u and v, two vectors, $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$, which are tangent, respectively, to the parametric curves $v =$ constant and $u =$ constant which pass through M. The vector product $\left[\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}\right]$ is normal to the surface S and is directed positively with respect to the smaller rotation from the progressive tangent to the u parametric curve (that is, that one along which v is constant) to the progressive tangent to the v parametric curve. If we have any vector field $\mathbf{A}$, whose values are given at all points of S, we may form the scalar product $\left(\mathbf{A}.\left[\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}\right]\right)$. When (x, y, z) in (A_x, A_y, A_z) are replaced by their expressions in terms of u and v, this becomes a function of the two independent variables u and v. Upon double integration with respect to u and v between limits determined by the surface S, we have a definite integral which is known as the integral of the vector $\mathbf{A}$ over the surface S or as the flux of the vector $\mathbf{A}$ through the surface S. This integral may be written out in the form

$$\int_S \left[A_x \frac{\partial(y, z)}{\partial(u, v)} + A_y \frac{\partial(z, x)}{\partial(u, v)} + A_z \frac{\partial(x, y)}{\partial(u, v)}\right] du\, dv. \qquad (13.3)$$

It readily follows from the formula for the change of the variables of integration in a double integral that (13.3) has a value independent of the choice of the parameters u and v. It is, accordingly, usually written in the form

$$\int_S [A_x d(y, z) + A_y d(z, x) + A_z d(x, y)], \qquad (13.4)$$

where no indication is given of the parameters to be used in evaluating it. Here $d(y, z)$ stands for $\frac{\partial(y, z)}{\partial(u, v)}\, du\, dv$, and so on. The magnitude of the vector $\left[\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}\right] du\, dv$ is denoted by dS and is called the element of area of the surface S. If $\boldsymbol{\nu} = (l, m, n)$ is that unit vector along the normal to S which is positively related to the smaller rotation from the u parametric curve to the v parametric curve, we may write

$$d(y, z) = l\,dS, \quad d(z, x) = m\,dS, \quad d(x, y) = n\,dS. \qquad (13.5)$$

Our surface integral then takes the form

$$\int_S (lA_x + mA_y + nA_z)dS, \quad \text{or} \int_S (\mathbf{A}.\boldsymbol{\nu})dS. \qquad (13.6)$$

The essential thing to notice is that the coefficients (A_x, A_y, A_z) of the surface integral (13.4) must present a vector if the integral is to have a value independent of the reference frame. We may, however, use the components of a tensor field $\mathfrak{A}$ to form an integral which is not a scalar but a vector. Thus we have

$$\begin{aligned}
&\int_S (lA_{xx} + mA_{xy} + nA_{xz})dS \\
&\qquad = \int_S A_{xx}\,d(y, z) + A_{xy}\,d(z, x) + A_{xz}\,d(x, y), \\
&\int_S (lA_{yx} + mA_{yy} + nA_{yz})dS \\
&\qquad = \int_S A_{yx}\,d(y, z) + A_{yy}\,d(z, x) + A_{yz}\,d(x, y), \qquad (13.7) \\
&\int_S (lA_{zx} + mA_{zy} + nA_{zz})dS \\
&\qquad = \int_S A_{zx}\,d(y, z) + A_{zy}\,d(z, x) + A_{zz}\,d(x, y)
\end{aligned}$$

as the three components of a vector (see § 11). The fact that the integrals present a vector when the integrands present a vector follows from the definition of integration as the limit of a summation.

c. Volume integrals. When the coördinates (x, y, z) of a variable point M are functions of three independent variables (α, β, γ), M traces out, as (α, β, γ) vary, a volume or region of space V. Through each point M of V there pass three curves which are obtained by putting β and γ, γ and α, α and β constant, respectively, in the expressions for (x, y, z) as functions of (α, β, γ). These curves are called coördinate curves, and the vector $\mathbf{r} = (x, y, z)$ yields, upon differentiation with respect to α, β, and γ, in turn, the three vectors $\frac{\partial \mathbf{r}}{\partial \alpha}, \frac{\partial \mathbf{r}}{\partial \beta}, \frac{\partial \mathbf{r}}{\partial \gamma}$, which are tangent, respectively, to the first, second, and third coördinate curves through M. The scalar triple product $\left(\frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial \mathbf{r}}{\partial \gamma}\right)$ is independent of the reference frame and gives the volume of the parallelepiped whose coterminous edges are representative segments of the vectors $\frac{\partial \mathbf{r}}{\partial \alpha}$, $\frac{\partial \mathbf{r}}{\partial \beta}$, and $\frac{\partial \mathbf{r}}{\partial \gamma}$ respectively. We may multiply this scalar triple product by a scalar point function $\phi(x, y, z)$; and, upon substituting for (x, y, z) their expressions as functions of (α, β, γ), the product $\phi \cdot \left(\frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial \mathbf{r}}{\partial \gamma}\right)$ becomes a function of (α, β, γ). We may then form the triple integral

$$\int_V \phi\left(\frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial \mathbf{r}}{\partial \gamma}\right) d\alpha\, d\beta\, d\gamma, \tag{13.8}$$

where the limits are determined by the volume V. Since $\left(\frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial \mathbf{r}}{\partial \gamma}\right)$ is equal to $\frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma)}$, it follows, from the rule for the change of variables of integration in a triple integral, that the integral (13.8) has a value independent of the choice of the curvilinear coördinates (α, β, γ). It is therefore written in the form

$$\int_V \phi\, d(x, y, z), \tag{13.9}$$

where no indication is given of the particular coördinates (α, β, γ) which are to be introduced for its evaluation. The integral (13.9) is known as the integral of the scalar point function ϕ throughout the volume V. Here

$$d(x, y, z) = \frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma)}\, d\alpha\, d\beta\, d\gamma$$

is known as the element of volume, and may be denoted by $d\tau$. If (α, β, γ) are the coördinates (x, y, z) themselves, we have $x = \alpha$, $y = \beta$, $z = \gamma$, so that $\frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma)} = 1$ and $d\tau = d(x, y, z) = dx\,dy\,dz$.

It is important to notice, again, that if the value of the volume integral (13.9) is to be independent of the reference frame, ϕ must be a scalar point function. We may, however, construct, from a given vector field $\mathbf{A}$, a volume integral which is not a scalar but a vector. Thus the three expressions

$$\int_V A_x\, d(x, y, z), \quad \int_V A_y\, d(x, y, z), \quad \int_V A_z\, d(x, y, z)$$

are the components of a vector which may be denoted by $\int_V \mathbf{A}\,d\tau$. The fact that the integrals just mentioned present a vector follows from the definition of an integral as the limit of a summation. In connection with this vector integral and that of (13.7) it may be well to point out that the possibility of adding up, in this way, vectors at different points M depends essentially on the fact that we have a fixed reference frame $OXYZ$, that is, one whose axes have the same directions at all points M. If we used, at each point M, a reference frame attached to it, — such, for example, as the frame furnished by a system of rectangular curvilinear coördinates, (α, β, γ), — the integrals $\int_V A_\alpha\,d\tau$, $\int_V A_\beta\,d\tau$, $\int_V A_\gamma\,d\tau$ would not, in general, present a vector. For if we consider two distinct points M_1 and M_2 and vectors $\mathbf{A}_1$ and $\mathbf{A}_2$, the vector $\mathbf{A}_1 + \mathbf{A}_2$ is not found in either of the two frames attached to M_1 and M_2, respectively, by adding the components $(A_\alpha)_1$ and $(A_\alpha)_2$, etc., since these frames have different orientations.

14. Green's lemma. There is an important connection, known as Green's lemma, between a surface integral over a closed surface and a certain derived integral extended throughout the volume bounded by the closed surface. In order to derive this lemma we shall use a Cartesian reference frame $OXYZ$ and shall consider a volume integral of the type $\int_V \frac{\partial f}{\partial x}\,d\tau$, where f is any point function with a continuous partial derivative $\frac{\partial f}{\partial x}$. Here $d\tau = dx\,dy\,dz$, and an integration with respect to x gives

$$\int_V \frac{\partial f}{\partial x}\,d\tau = \int_{S'} f\Big|_1^2\,dy\,dz.$$

The integration with respect to y and z is over the projection S', on the yz plane, of the closed surface which bounds the volume V; and $f\big|_1^2$ represents the difference between the sum of the values of f at the even points of intersection of the surface S with a straight line parallel to the x-axis, and the sum of the values of f at the odd points of intersection of the surface with the same straight line. Now, by (13.5), $dy\,dz = l\,dS$, where (l, m, n) is the unit vector along the direction of the normal which makes an acute angle with the x-axis; for here y and z are the two parameters on the surface S, and the parametric curves on S are the plane curves of intersection of S with planes parallel to the z and y coördinate planes respectively. The normal which is positively related to a rotation from the y parametric curve to the z parametric curve makes an acute angle with the x-axis. If we adopt the notation (l, m, n) for the presentation of the outward-drawn unit vector $\boldsymbol{\nu}$ along the normal, we have $dy\,dz = l\,dS$ at the even points of intersection of S with a line parallel to the x-axis, and $dy\,dz = -l\,dS$ at the odd points of intersection; so that

$$\int_{S'} f\Big|_1^2 dy\,dz = \int_S fl\,dS,$$

where the integral on the right is extended over the entire closed surface S. We may derive similar results for integrals of the type $\int_V \frac{\partial g}{\partial y} dy$ and $\int_V \frac{\partial h}{\partial z} dz$. The three results

$$\int_V \frac{\partial f}{\partial x} d\tau = \int_S lf\,dS, \quad \int_V \frac{\partial g}{\partial y} d\tau = \int_S mg\,dS,$$
$$\int_V \frac{\partial h}{\partial z} d\tau = \int_S nh\,dS, \tag{14.1}$$

where (l, m, n) presents the unit vector along the *outward*-drawn normal to the surface S, constitute Green's lemma. (By "outward" is meant away from the volume V.)

Applying the lemma to the volume integral $\int_V \operatorname{div} \mathbf{A}\,d\tau$, where $\mathbf{A}$ is a vector field with continuous partial derivatives throughout V, we have the result

$$\int_V \operatorname{div} \mathbf{A}\,d\tau = \int_S (lA_x + mA_y + nA_z)dS = \int_S (\mathbf{A}.\boldsymbol{\nu})dS. \tag{14.2}$$

This result is frequently referred to as Gauss's theorem. The scalar product $(\mathbf{A}.\boldsymbol{\nu})$ is the resolved part of $\mathbf{A}$ along the outward-drawn normal to S; and, denoting this by A_n, we may rewrite (14.2) in the form

$$\int_V \operatorname{div} \mathbf{A}\,d\tau = \int_S A_n\,dS. \tag{14.2 a}$$

The equation gives the mean value of div $\mathbf{A}$ throughout the volume V as equal to $\int_S A_n\,dS$ divided by the inclosed volume. If the volume V is decreased indefinitely in all its dimensions, but in such a manner as always to contain a fixed point, we have for the value of div **A** at this point the limit

$$\operatorname{div} \mathbf{A} = \underset{V \to 0}{\mathrm{Lt}} \left[\frac{\int_S A_n \, dS}{V} \right].$$

The numerator here is the flux of **A** across the closed surface S surrounding the point in question, and it is from this relation that the name "divergence" is derived. It may not be out of place to observe that the relation just written may be used as a definition of div **A**. This definition is of wider application than that given previously, as, according to it, the divergence may well exist when some of the partial derivatives, $\frac{\partial A_x}{\partial x}$ etc., do not exist. In particular, if **A** is the gradient of a scalar point function ϕ, $(\mathbf{A}.\nu)$, or A_n, is the normal directional derivative $\frac{d\phi}{dn}$; and we have

$$\int_V \Delta_2 \phi \, d\tau = \int_S \frac{d\phi}{dn} \, dS. \qquad (14.3)$$

It is assumed that the second partial derivatives $\frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^2 \phi}{\partial y^2}, \frac{\partial^2 \phi}{\partial z^2}$ are continuous throughout the region of integration. If there are certain singular points in the volume at which these derivatives become infinite, these points may be excluded from the region of integration by drawing small closed surfaces around each of them. The bounding surface S of the region of integration will then consist of the original surface together with the surfaces which have thus been introduced in order to exclude the singular points.

We may derive from (14.2) a slightly more general result than (14.3) if we write $\mathbf{A} = \psi$ gard ϕ, where both ψ and ϕ are scalar point functions. We find

$$\int_V [\psi \Delta_2 \phi + (\mathbf{grad}\, \psi . \mathbf{grad}\ \phi)] d\tau = \int_S \psi \frac{d\phi}{dn} \, dS. \qquad (14.4)$$

This result is often referred to as Green's first theorem. Upon interchanging the rôles of ϕ and ψ in (14.4) and subtracting the result from (14.4), we obtain

$$\int_V (\psi \Delta_2 \phi - \phi \Delta_2 \psi) d\tau = \int_S \left(\psi \frac{d\phi}{dn} - \phi \frac{d\psi}{dn} \right) dS, \qquad (14.5)$$

a result which is sometimes referred to as Green's second theorem.

If the scalar point functions ϕ and ψ in (14.4) are identical, we have the result

$$\int_V (\phi\Delta_2\phi + \Delta_1\phi)d\tau = \int_S \phi\frac{d\phi}{dn}dS. \tag{14.6}$$

It follows from (14.6) that a scalar point function ϕ which has the properties that $\Delta_2\phi = 0$ throughout a volume V, and that either $\phi = 0$ or $\frac{d\phi}{dn} = 0$ over the closed surface S bounding V, must satisfy the equation

$$\int_V \Delta_1\phi\, d\tau = 0. \tag{14.7}$$

But $\Delta_1\phi$, being the square of the magnitude of **grad** ϕ, cannot be negative; and as $d\tau$ is essentially positive, we see, from (14.7), that $\Delta_1\phi$ must be identically zero throughout the volume V. For the integral of a definitely positive (in the sense of non-negative) continuous function over any region cannot be zero unless the integrand is everywhere zero. To say that $\Delta_1\phi$ is zero is to say that **grad** ϕ is zero or that ϕ is constant throughout V. In particular, if ϕ is considered to be zero on S, it must be zero throughout V. More generally, if we are given two point functions W_1 and W_2 having the same values for Δ_2W_1 and Δ_2W_2 at each point of V, and the same values for $\frac{dW_1}{dn}$ and $\frac{dW_2}{dn}$ at each point of S, the difference between W_1 and W_2 must be constant throughout V. For this difference has the properties assigned to the scalar point function ϕ above. Similarly, if Δ_2W_1 and Δ_2W_2 have the same values at each point of V, and W_1 and W_2 have the same values at each point of S, then W_1 and W_2 must have the same values at each point of V. These results constitute what are known as *uniqueness theorems.*

15. Stokes's theorem. If we have a plane closed curve C bounding a plane area S, we have, by a method exactly similar to that given above for space, Green's lemma for the plane. Thus, if f and g are any two functions of the coördinates (x, y) of a variable point in the plane, we have

$$\int_S \frac{\partial f}{\partial x}dS = \oint_C lf\,ds, \qquad \int_S \frac{\partial g}{\partial y}dS = \oint_C mg\,ds, \tag{15.1}$$

where ds is an element of length of arc along C, and dS is an element of area of S. Here l and m are the direction cosines of the outward-drawn normal ν to the curve C, and the sense of integration around C is such as to leave the area S on the left. In addition, the axes of x and y are supposed to be so chosen that the sense of the rotation from x to y is the same as the sense of the integration around C. At every point of the area S the functions f and g are supposed to possess continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial y}$ respectively.

The progressive tangent to the curve C is ahead of the outward-drawn normal ν by a right angle; so that we have

$$\begin{aligned} \frac{dx}{ds} &= \cos\left(\widehat{\nu x} + \frac{\pi}{2}\right) = \cos\,(\widehat{\nu y} + \pi) = -\,m, \\ \frac{dy}{ds} &= \cos\left(\widehat{\nu y} + \frac{\pi}{2}\right) = \cos\,(\widehat{\nu x}) = l, \end{aligned} \tag{15.2}$$

where $\widehat{\nu x}$ and $\widehat{\nu y}$ denote, respectively, the angles from the x and y axes to the outward-drawn normal.

We may rewrite the results (15.1) in the form

$$\int_S \frac{\partial f}{\partial x}\,dS = \oint_C f\,dy, \qquad \int_S \frac{\partial g}{\partial y}\,dS = -\oint_C g\,dx. \tag{15.3}$$

Upon combining these results, and writing $-\,u$ for g and v for f, we have

$$\oint_C u\,dx + v\,dy = \int_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dS = \int_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) d(x, y). \tag{15.4}$$

Equation (15.4) is Stokes's lemma for the plane. In order to extend it to any closed curve in space, let us first consider a triangle whose vertices are any three points (A, B, C) in space. It is always possible to find a point O such that the three lines OA, OB, and OC may be chosen as the x, y, and z axes of a right-handed Cartesian reference frame. The point O is, in fact, one of the two intersections of the three spheres which have BC, CA, and AB, respectively, as diameters. Let (u, v, w) be the presentation of a vector in this frame, and consider the line integral

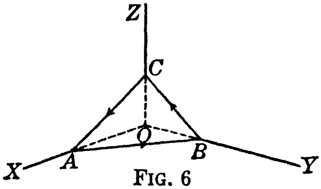

FIG. 6

$$\oint_{ABC} (u\,dx + v\,dy + w\,dz) \tag{15.5}$$

taken around the closed path ABC formed by the three sides of the triangle ABC. This path may be replaced by the sequence of paths BOC, COA, AOB; for the line integrals over the new paths BO and OB, OC and CO, OA and AO mutually cancel. The path BOC lies in the yz plane, so that along it $dx = 0$, and our line integral (15.5) becomes $\oint_{BOC} (v\,dy + w\,dz)$. This is, by (15.4), equal to

$$\int \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) d(y, z),$$

the integration being extended over the area of the triangle BOC. If dS is an element of area of the triangle ABC, and (l, m, n) are the direction cosines of the outward-drawn normal to this triangle (that is, the normal whose direction is positively related to the sense ABC of integration around the triangle), we have, by (13.5), $d(y, z) = l\,dS$. So we may write

$$\oint_{BOC} (u\,dx + v\,dy + w\,dz) = \int_S l\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) dS.$$

The surface integral is now extended over the area of the triangle ABC. Proceeding similarly with the line integrals along the closed paths COA and AOB, and combining our results, we find

$$\oint_{ABC} (u\,dx + v\,dy + w\,dz) = \int_S \left[l\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + m\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + n\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] dS. \tag{15.6}$$

Denoting the vector whose presentation in the frame $OABC$ is (u, v, w) by **A**, we may write (15.6) in the vector form

$$\begin{aligned}\oint_{ABC} (\mathbf{A}.\mathbf{ds}) &= \int_S (\mathbf{curl\ A}.\mathbf{dS}) \\ &= \int_S (\mathbf{curl\ A}.\boldsymbol{\nu})\,dS = \int_S (\mathbf{curl\ A})_\nu\,dS, \end{aligned} \tag{15.7}$$

where **ds** is the differential vector along the closed path ABC, and **dS** is the differential surface vector whose components are $d(y, z)$, $d(z, x)$, $d(x, y)$.

The equation (15.7) says that the average value of $(\mathbf{curl\ A})_\nu$ over the area of the triangle ABC is $\oint_{ABC} (\mathbf{A}.\mathbf{ds})$ divided by the area of the triangle. As the triangle is decreased indefinitely in all its dimensions, but so as always to contain a certain point P, we obtain the following limit definition of $(\mathbf{curl\,A})_\nu$ at the point P:

$$(\mathbf{curl\,A})_\nu = \underset{\Delta \to 0}{\mathrm{Lt}} \left[\frac{\oint (\mathbf{A}.\mathbf{ds})}{\Delta} \right],$$

where Δ is the area of the triangle ABC. As in the case of the divergence of a vector field, this definition is of wider application than that given previously. The symbol **rot A** is sometimes used instead of **curl A**. Both symbols have a hydrodynamic origin.

Let us now suppose that we have any closed curve C bounding an arbitrary surface S. We may regard S as the limit approached by a figure with triangular faces as each side of every one of these triangular faces is decreased indefinitely. Applying (15.7) to each of these triangular faces, and observing that the only sides of the triangles which are not traced twice in opposite directions are those which join contiguous points on the bounding curve C, we obtain by this limiting process the result

$$\begin{aligned} \oint_C (\mathbf{A}.\mathbf{ds}) &= \int_S (\mathbf{curl\,A}.\mathbf{dS}) \\ &= \int_S (\mathbf{curl\,A}.\nu)dS = \int_S (\mathbf{curl\,A})_\nu\, dS. \qquad (15.8) \end{aligned}$$

This is known as Stokes's theorem. It was first communicated to Stokes by William Thomson, afterwards Lord Kelvin, and was then proposed by Stokes in the Smith prize examination in 1854. It may be conveniently stated in words as follows:

The line integral of a vector around any closed curve is equal to the flux of the curl of that vector through any surface bounded by the curve. The direction of the flux is positively related to the sense of integration around the curve.

Stokes's theorem may be proved in the following manner, which does not use the limiting process mentioned above. This proof is interesting since it can be extended at once to spaces of any number of dimensions and of no special metrical character. The extended form of Stokes's theorem thus obtained will be very useful when we come to consider dynamic systems and their representative phase spaces.

Adopting the parametric representation of points on a surface, the coördinates (x, y, z) of each point on the surface are functions of two independent variables, u and v. We may regard these variables (u, v) as the Cartesian coördinates of a point on a plane; and so the equations of the surface,

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \tag{15.9}$$

set up a correspondence between the points $P = (x, y, z)$ of the surface and the points $P' = (u, v)$ of the plane whose coördinates are connected with those of P by the equations (15.9). The surface S bounded by the closed curve C will have corresponding to it in this way a portion S' of the plane bounded by a closed curve C'. Upon substituting for (x, y, z) their expressions in terms of (u, v) given by the equations (15.9), any line integral of the type

$$\oint_C (X\,dx + Y\,dy + Z\,dz), \tag{15.10}$$

where (X, Y, Z) are functions of (x, y, z), will transform into a line integral over the closed plane curve C'. Thus we have an equation of the type

$$\oint_C (X\,dx + Y\,dy + Z\,dz) = \oint_{C'} (U\,du + V\,dv), \tag{15.11}$$

where

$$U = X\frac{\partial x}{\partial u} + Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} \quad \text{and} \quad V = X\frac{\partial x}{\partial v} + Y\frac{\partial y}{\partial v} + Z\frac{\partial z}{\partial v}, \tag{15.12}$$

the quantities (X, Y, Z) being expressed in terms of u and v. Applying the result (15.4) to the integral $\oint_{C'} (U\,du + V\,dv)$, extended along the closed plane curve C', we have

$$\oint_{C'} (U\,du + V\,dv) = \int_{S'} \left(\frac{\partial V}{\partial u} - \frac{\partial U}{\partial v}\right) d(u, v). \tag{15.13}$$

From (15.12) we find

$$\frac{\partial V}{\partial u} = X\frac{\partial^2 x}{\partial u\,\partial v} + Y\frac{\partial^2 y}{\partial u\,\partial v} + Z\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial X}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial Y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial Z}{\partial u}\frac{\partial z}{\partial v},$$

$$\frac{\partial U}{\partial v} = X\frac{\partial^2 x}{\partial u\,\partial v} + Y\frac{\partial^2 y}{\partial u\,\partial v} + Z\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial X}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial Y}{\partial v}\frac{\partial y}{\partial u} + \frac{\partial Z}{\partial v}\frac{\partial z}{\partial u},$$

so that

$$\frac{\partial V}{\partial u} - \frac{\partial U}{\partial v} = \frac{\partial(X, x)}{\partial(u, v)} + \frac{\partial(Y, y)}{\partial(u, v)} + \frac{\partial(Z, z)}{\partial(u, v)}.$$

Upon substituting for $\frac{\partial X}{\partial u}$ its value $\frac{\partial X}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial u}$ found by composite differentiation, we have

$$\frac{\partial(X, x)}{\partial(u, v)} = \frac{\partial X}{\partial z}\cdot\frac{\partial(z, x)}{\partial(u, v)} - \frac{\partial X}{\partial y}\cdot\frac{\partial(x, y)}{\partial(u, v)}.$$

There are similar expressions for $\dfrac{\partial(Y, y)}{\partial(u, v)}$ and $\dfrac{\partial(Z, z)}{\partial(u, v)}$ respectively. Upon combining these results, we have

$$\frac{\partial V}{\partial u} - \frac{\partial U}{\partial v} = \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\frac{\partial(y, z)}{\partial(u, v)} + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\frac{\partial(z, x)}{\partial(u, v)} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\frac{\partial(x, y)}{\partial(u, v)},$$

so that

$$\begin{aligned}\oint_{C'}(U\,du + V\,dv) &= \int_{S'}\left[\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\frac{\partial(y, z)}{\partial(u, v)} + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\frac{\partial(z, x)}{\partial(u, v)} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\frac{\partial(x, y)}{\partial(u, v)}\right]d(u, v)\\ &= \int_S\left[l\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + m\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + n\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\right]dS. \qquad \text{[See 13.5]}\end{aligned}$$

The application of (15.11) gives us

$$\oint_C(X\,dx + Y\,dy + Z\,dz) = \int_S\left[l\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + m\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + n\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\right]dS, \qquad (15.14)$$

which is Stokes's theorem.

It is evident that this mode of proof is applicable to a space of any number of dimensions. Thus, if we have a line integral of the type

$$\oint_C(X_1\,dx_1 + X_2\,dx_2 + \cdots + X_n\,dx_n)$$

over a closed curve in such a space, it is equivalent to a surface integral

$$\int_S \sum_{\substack{1\\ r<s}}^{n} X_{rs}\,d(x_r, x_s), \qquad (15.15)$$

extended over any surface S bounded by the closed curve C. Here the coefficients X_{rs} of the surface integral are defined by the equations

$$X_{rs} = \frac{\partial X_s}{\partial x_r} - \frac{\partial X_r}{\partial x_s}. \qquad (15.16)$$

There are $\dfrac{n(n-1)}{2}$ (that is, the number of combinations of n things two at a time) terms in the integrand of the surface integral.*

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* For this extension of Stokes's theorem, and for a discussion of the connection between Stokes's theorem and Green's lemma and Gauss's theorem, reference may be made to a paper entitled "The Absolute Significance of Maxwell's Equations," *Physical Review*, 17, 73 (1921).

16. Orthogonal curvilinear coördinates. If we apply Green's lemma to the vector integral formed with the vector of (12.2), we have the equations

$$\begin{aligned}
\int_V\left(\frac{\partial A_{xx}}{\partial x}+\frac{\partial A_{xy}}{\partial y}+\frac{\partial A_{xz}}{\partial z}\right)d\tau &= \int_S(lA_{xx}+mA_{xy}+nA_{xz})dS,\\
\int_V\left(\frac{\partial A_{yx}}{\partial x}+\frac{\partial A_{yy}}{\partial y}+\frac{\partial A_{yz}}{\partial z}\right)d\tau &= \int_S(lA_{yx}+mA_{yy}+nA_{yz})dS, \qquad (16.1)\\
\int_V\left(\frac{\partial A_{zx}}{\partial x}+\frac{\partial A_{zy}}{\partial y}+\frac{\partial A_{zz}}{\partial z}\right)d\tau &= \int_S(lA_{zx}+mA_{zy}+nA_{zz})dS.
\end{aligned}$$

We now proceed to find the presentation in a frame of variable orientation of the vector which occurs in the volume integral on the left, as this presentation is very useful in problems on the dynamics of continuous media. Let us consider the varying reference frames each of which is attached to a point (α, β, γ) of a system of curvilinear coördinates. Introducing, as in §6, the notations (x_1, x_2, x_3) and $(\alpha_1, \alpha_2, \alpha_3)$ for (x, y, z) and (α, β, γ) respectively, and denoting the components of the tensor $\mathfrak{A}$ in the frame attached to the point (α, β, γ) and in the fixed frame by

$$\begin{array}{ccc}
\widehat{11} & \widehat{12} & \widehat{13}\\
\widehat{21} & \widehat{22} & \widehat{23}\\
\widehat{31} & \widehat{32} & \widehat{33}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A_{11} & A_{12} & A_{13}\\
A_{21} & A_{22} & A_{23}\\
A_{31} & A_{32} & A_{33}
\end{array}$$

respectively, we have, from the equations (8.4), the general formula

$$A_{rs}=\sum_{\substack{p,q\\1}}^{3} c_r{}^p c_s{}^q\,\widehat{pq}. \tag{16.2}$$

Since p and q take independently all the values 1, 2, 3, there are nine terms in the double summation; and since r and s take independently the values 1, 2, 3, there are nine equations of the type (16.2). Taking the fixed Cartesian frame parallel to the frame which is attached to some particular point $(\alpha_0, \beta_0, \gamma_0)$ in which we are interested, we have, from (6.15) and (16.2),

$$(A_{rs})_0=(\widehat{rs})_0. \tag{16.3}$$

The first component, $\frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{xy}}{\partial y} + \frac{\partial A_{xz}}{\partial z}$, of the vector in which we are interested takes the form $\sum_{m=1}^{3} \frac{\partial A_{1m}}{\partial x_m}$; and this may be replaced, on using (16.2), by $\sum_{m,p,q \atop 1}^{3} \frac{\partial}{\partial x_m} (c_1{}^p\, c_m{}^q\, \widehat{pq})$. Upon substituting the values at (α_0, β_0, γ_0), we obtain

$$\left(\frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{xy}}{\partial y} + \frac{\partial A_{xz}}{\partial z}\right)_0 = \left[\sum_{m,p,q \atop 1}^{3} \frac{\partial}{\partial x_m} (c_1{}^p\, c_m{}^q\, \widehat{pq})\right]_0.$$

This becomes, on performing the differentiations and using (6.15),

$$\left[\sum_{m=1}^{3} \frac{\partial}{\partial x_m} (\widehat{1\,m}) + \sum_{m,\,p, \atop 1}^{3} \widehat{pm} \frac{\partial}{\partial x_m} (c_1{}^p) + \sum_{m,\,q \atop 1}^{3} \widehat{1\,q} \frac{\partial}{\partial x_m} (c_m{}^q)\right]_0$$

We now make use of the results of (6.20), (6.23), (6.28), and (6.29), and the expression just written becomes

$$\left[\sum_{m=1}^{3} \frac{\partial}{\partial x_m} (\widehat{1\,m}) + \widehat{21} \frac{\partial}{\partial x_1} c_1{}^2 + \widehat{31} \frac{\partial}{\partial x_1} c_1{}^3 + \widehat{22} \frac{\partial}{\partial x_2} c_1{}^2 + \widehat{33} \frac{\partial}{\partial x_3} c_1{}^3 \right.$$
$$+ \widehat{12} \frac{\partial}{\partial x_1} c_1{}^2 + \widehat{13} \frac{\partial}{\partial x_1} c_1{}^3 + \widehat{11} \frac{\partial}{\partial x_2} c_2{}^1 + \widehat{13} \frac{\partial}{\partial x_2} c_2{}^3$$
$$\left. + \widehat{11} \frac{\partial}{\partial x_3} c_3{}^1 + \widehat{12} \frac{\partial}{\partial x_3} c_3{}^2 \right]_0.$$

On writing $\left(\frac{\partial}{\partial x_m}\right)_0 = \left(h_m \frac{\partial}{\partial \alpha_m}\right)_0$ (from (6.18)) and using (6.29), we may put this expression in the form

$$\left\{h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha_1}\left(\frac{\widehat{11}}{h_2 h_3}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{\widehat{12}}{h_3 h_1}\right) + \frac{\partial}{\partial \alpha_3}\left(\frac{\widehat{13}}{h_1 h_2}\right)\right] + \widehat{21}\, h_1 h_2 \frac{\partial}{\partial \alpha_2}\left(\frac{1}{h_1}\right) \right.$$
$$+ \widehat{31}\, h_1 h_3 \frac{\partial}{\partial \alpha_3}\left(\frac{1}{h_1}\right) - \widehat{22}\, h_1 h_2 \frac{\partial}{\partial \alpha_1}\left(\frac{1}{h_2}\right)$$
$$\left. - \widehat{33}\, h_1 h_3 \frac{\partial}{\partial \alpha_1}\left(\frac{1}{h_3}\right)\right\}_0.$$

Since the point $P_0 = (\alpha_0, \beta_0, \gamma_0)$ is an arbitrary point, we may drop the zero subscript: and we find that the presentation, in the frame attached to the point (α, β, γ), of the vector field given in (12.2) is

$$\begin{aligned}
&h_1h_2h_3\left[\frac{\partial}{\partial\alpha}\left(\frac{\widehat{\alpha\alpha}}{h_2h_3}\right)+\frac{\partial}{\partial\beta}\left(\frac{\widehat{\alpha\beta}}{h_3h_1}\right)+\frac{\partial}{\partial\gamma}\left(\frac{\widehat{\alpha\gamma}}{h_1h_2}\right)\right]\\
&\qquad+\widehat{\beta\alpha}\,h_1h_2\frac{\partial}{\partial\beta}\frac{1}{h_1}+\widehat{\gamma\alpha}\,h_1h_3\frac{\partial}{\partial\gamma}\frac{1}{h_1}\\
&\qquad-\widehat{\beta\beta}\,h_1h_2\frac{\partial}{\partial\alpha}\frac{1}{h_2}-\widehat{\gamma\gamma}\,h_1h_3\frac{\partial}{\partial\alpha}\frac{1}{h_3},\\
&h_1h_2h_3\left[\frac{\partial}{\partial\alpha}\left(\frac{\widehat{\beta\alpha}}{h_2h_3}\right)+\frac{\partial}{\partial\beta}\left(\frac{\widehat{\beta\beta}}{h_3h_1}\right)+\frac{\partial}{\partial\gamma}\left(\frac{\widehat{\beta\gamma}}{h_1h_2}\right)\right]\\
&\qquad+\widehat{\alpha\beta}\,h_1h_2\frac{\partial}{\partial\alpha}\frac{1}{h_2}+\widehat{\gamma\beta}\,h_2h_3\frac{\partial}{\partial\gamma}\frac{1}{h_2}\\
&\qquad-\widehat{\alpha\alpha}\,h_1h_2\frac{\partial}{\partial\beta}\frac{1}{h_1}-\widehat{\gamma\gamma}\,h_2h_3\frac{\partial}{\partial\beta}\frac{1}{h_3},\\
&h_1h_2h_3\left[\frac{\partial}{\partial\alpha}\left(\frac{\widehat{\gamma\alpha}}{h_2h_3}\right)+\frac{\partial}{\partial\beta}\left(\frac{\widehat{\gamma\beta}}{h_3h_1}\right)+\frac{\partial}{\partial\gamma}\left(\frac{\widehat{\gamma\gamma}}{h_1h_2}\right)\right]\\
&\qquad+\widehat{\alpha\gamma}\,h_1h_3\frac{\partial}{\partial\alpha}\frac{1}{h_3}+\widehat{\beta\gamma}\,h_2h_3\frac{\partial}{\partial\beta}\frac{1}{h_3}\\
&\qquad-\widehat{\alpha\alpha}\,h_1h_3\frac{\partial}{\partial\gamma}\frac{1}{h_1}-\widehat{\beta\beta}\,h_2h_3\frac{\partial}{\partial\gamma}\frac{1}{h_2}.
\end{aligned}\tag{16.4}$$

Here the presentation of the tensor $\mathfrak{A}$ in the frame attached to (α, β, γ) is denoted for convenience by

$$\begin{matrix}
\widehat{\alpha\alpha} & \widehat{\alpha\beta} & \widehat{\alpha\gamma}\\
\widehat{\beta\alpha} & \widehat{\beta\beta} & \widehat{\beta\gamma}\\
\widehat{\gamma\alpha} & \widehat{\gamma\beta} & \widehat{\gamma\gamma}
\end{matrix}\tag{16.5}$$

As special cases we give the form of (16.4) for cylindrical and space polar coördinates.

For cylindrical coördinates we have $h_1 = 1$, $h_2 = \dfrac{1}{\rho}$, $h_3 = 1$, and we find as the presentation in cylindrical coördinates

$$\begin{aligned}
&\frac{\partial}{\partial\rho}\widehat{\rho\rho}+\frac{1}{\rho}\widehat{\rho\rho}+\frac{1}{\rho}\frac{\partial}{\partial\phi}\widehat{\rho\phi}+\frac{\partial}{\partial z}\widehat{\rho z}-\frac{1}{\rho}\widehat{\phi\phi},\\
&\frac{\partial}{\partial\rho}\widehat{\phi\rho}+\frac{1}{\rho}(\widehat{\phi\rho}+\widehat{\rho\phi})+\frac{1}{\rho}\frac{\partial}{\partial\phi}\widehat{\phi\phi}+\frac{\partial}{\partial z}\widehat{\phi z},\\
&\frac{\partial}{\partial\rho}\widehat{z\rho}+\frac{1}{\rho}\widehat{z\rho}+\frac{1}{\rho}\frac{\partial}{\partial\phi}\widehat{z\phi}+\frac{\partial}{\partial z}\widehat{zz}.
\end{aligned}\tag{16.6}$$

For space polar coördinates $h_1 = 1$, $h_2 = \frac{1}{r}$, $h_3 = \frac{1}{r \sin \theta}$. The presentation in this system of coördinates is therefore

$$\begin{aligned}
&\frac{\partial}{\partial r} \widehat{rr} + \frac{2}{r} \widehat{rr} + \frac{1}{r} \frac{\partial}{\partial \theta} \widehat{r\theta} + \frac{\cot \theta}{r} \widehat{r\theta} \\
&\qquad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \widehat{r\phi} - \frac{1}{r} (\widehat{\theta\theta} + \widehat{\phi\phi}), \\
&\frac{\partial}{\partial r} \widehat{\theta r} + \frac{2}{r} \widehat{\theta r} + \frac{1}{r} \frac{\partial}{\partial \theta} \widehat{\theta\theta} + \frac{\cot \theta}{r} (\widehat{\theta\theta} - \widehat{\phi\phi}) \\
&\qquad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \widehat{\theta\phi} + \frac{\widehat{r\theta}}{r}, \\
&\frac{\partial}{\partial r} \widehat{\phi r} + \frac{2}{r} \widehat{\phi r} + \frac{1}{r} \frac{\partial}{\partial \theta} \widehat{\phi\theta} + \frac{\cot \theta}{r} (\widehat{\phi\theta} + \widehat{\theta\phi}) \\
&\qquad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \widehat{\phi\phi} + \frac{1}{r} \widehat{r\phi}.
\end{aligned} \tag{16.7}$$

When the tensor $\mathfrak{A}$ is symmetric, these rather complicated expressions become somewhat simplified.

17. Localized vectors. There are certain physical quantities which require for their full description a magnitude and a direction, and, in addition, the position in space of a straight line having this direction. Thus, let us consider the rotation of a door on its axis, that is, the line joining its hinges. In addition to knowing the amount of the rotation and the direction of the axis — that is, the sense of the rotation — it is necessary to know the position of the axis. We may picture a physical quantity such as this rotation by a vector located on the specified line. It is immaterial what particular point of the line is taken as the initial point of the representative segment of the vector, and all representative segments may be obtained from any one of them by merely sliding it along the axis. Such a physical quantity is known as a *localized vector* quantity. Any one of its representative segments is known as a sliding segment or a localized vector. The axis along which the representative segment may slide is called the *axis* of the localized vector. As in the case of nonlocalized vectors, we shall frequently use the term "localized vector" to designate either the physical quantity or the representative sliding segment.

Let us agree to denote points by small letters, such as *a*, *b*, *c*, etc. By $\mathbf{A}_a$ shall be meant a localized vector whose axis passes through the point a. By $\mathbf{A}$ will be understood the nonlocalized vector of which any one of the representative segments of $\mathbf{A}_a$ is a representative segment. Denoting by $\mathbf{r}_{ab}$ the nonlocalized vector of which the segment $\overrightarrow{ab}$ is a representative segment, the vector product

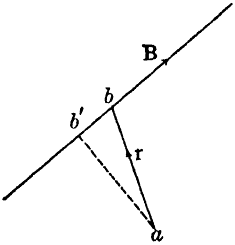

FIG. 7

$$[\mathbf{r}_{ab}.\mathbf{B}] \tag{17.1}$$

is called the vector moment of the localized vector $\mathbf{B}_b$ about the point a. This vector moment is independent of the position of the point b on the axis of the localized vector $\mathbf{B}_b$. For if b' be a different point on this axis, we have

$$\mathbf{r}_{bb'} = m\mathbf{B},$$

where m is a scalar quantity. So

$$\mathbf{r}_{ab'} = \mathbf{r}_{ab} + \mathbf{r}_{bb'} = \mathbf{r}_{ab} + m\mathbf{B}.$$

Hence, by (3.9), $\quad [\mathbf{r}_{ab'}.\mathbf{B}] = [\mathbf{r}_{ab}.\mathbf{B}].$

Let us now suppose that we have two points a and b and a localized vector $\mathbf{C}_c$. Then

$$[\mathbf{r}_{ac}.\mathbf{C}] = [\mathbf{r}_{ab}.\mathbf{C}] + [\mathbf{r}_{bc}.\mathbf{C}]. \tag{17.2}$$

This equation shows that the vector moment of any localized vector $\mathbf{C}_c$ about any point a is equal to the vector moment of the same localized vector $\mathbf{C}_c$ about b, plus the vector moment of the localized vector $\mathbf{C}_b$ about a. Hence, in order to know the vector moments about all points in space of a localized vector $\mathbf{C}_c$, it is sufficient to know the nonlocalized vector $\mathbf{C}$ and the vector moment

$$\mathbf{G} = [\mathbf{r}_{oc}.\mathbf{C}] \tag{17.3}$$

of $\mathbf{C}_c$ about some convenient origin o. The vector moment about any point a in space is then given by

$$\mathbf{G} + [\mathbf{r}_{ao}.\mathbf{C}] \quad \text{or} \quad \mathbf{G} - [\mathbf{r}_{oa}.\mathbf{C}]. \tag{17.4}$$

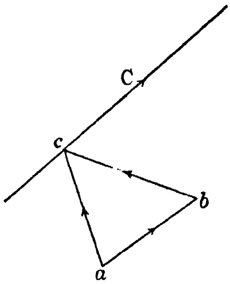

FIG. 8

If, in a Cartesian frame with origin at o, $\mathbf{C}$ has the presentation (X, Y, Z) and $\mathbf{G}$ has the presentation (L, M, N), the six quantities (X, Y, Z, L, M, N) are called the coördinates of the sliding segment which represents the localized vector $\mathbf{C}_c$. It follows at once, from (17.3), that $(\mathbf{C}.\mathbf{G}) = 0$, so that the six coördinates of the sliding segment are connected by the relation

$$XL + YM + ZN = 0. \qquad (17.5)$$

If (x, y, z) are the coördinates of any point c on the axis of $\mathbf{C}_c$, we have, from (17.3), the three equations

$$L = yZ - zY, \quad M = zX - xZ, \quad N = xY - yX. \qquad (17.6)$$

On account of (17.5) these three planes have a common line and not merely a common point. The axis of the localized vector whose coördinates are (X, Y, Z, L, M, N) lies along the common line of the three planes (17.6).

If we resolve the vector moment $[\mathbf{r}_{ac}.\mathbf{C}]$ of a localized vector $\mathbf{C}_c$ about any point a in any direction, the resulting scalar quantity is called the moment of $\mathbf{C}_c$ about the axis through a having this direction. If $\mathbf{A}_1$ is a unit nonlocalized vector having the direction of this axis, we have, for the moment of $\mathbf{C}_c$ about the axis, the scalar triple product

$$(\mathbf{A}_1.\mathbf{r}_{ac}.\mathbf{C}). \qquad (17.7)$$

If a' is any other point on the axis through a, we have $\mathbf{r}_{aa'} = m\mathbf{A}_1$, where m is a scalar. Consequently $\mathbf{r}_{ac} = \mathbf{r}_{a'c} + m\mathbf{A}_1$. This shows, on using (3.12), that the moment of $\mathbf{C}_c$ about the axis through a is independent of the position of a on the axis. We have already seen that it is also independent of the position of the point c on the axis of $\mathbf{C}_c$.

If $\mathbf{A}_a$ and $\mathbf{B}_b$ are any two localized vectors, the scalar expression

$$M_{12} = (\mathbf{A}.\mathbf{r}_{ab}.\mathbf{B}) \qquad (17.8)$$

is called the relative moment of the localized vector $\mathbf{B}_b$ and the localized vector $\mathbf{A}_a$. It is independent of the positions of a and b on the axes of $\mathbf{A}_a$ and $\mathbf{B}_b$ respectively. Furthermore, it is independent of the order in which the localized vectors are taken; for

$$(\mathbf{B}.\mathbf{r}_{ba}.\mathbf{A}) = (\mathbf{B}.\mathbf{A}.\mathbf{r}_{ab}) = (\mathbf{A}.\mathbf{r}_{ab}.\mathbf{B}).$$

Let $(X_1, Y_1, Z_1, L_1, M_1, N_1)$ be the coördinates of the first sliding segment, and $(X_2, Y_2, Z_2, L_2, M_2, N_2)$ those of the second. Writing $\mathbf{r}_{ab}=\mathbf{r}_{ao}+\mathbf{r}_{ob}$ and $[\mathbf{r}_{ob}.\mathbf{B}]=\mathbf{G}_2$, as also $[\mathbf{A}.\mathbf{r}_{ao}] = [\mathbf{r}_{oa}.\mathbf{A}] = \mathbf{G}_1$, we have

$$\begin{aligned} M_{12} &= (\mathbf{A}.\mathbf{G}_2) + (\mathbf{B}.\mathbf{G}_1) \\ &= X_1L_2 + Y_1M_2 + Z_1N_2 + X_2L_1 + Y_2M_1 + Z_2N_1. \qquad (17.9) \end{aligned}$$

Here (L_1, M_1, N_1), for example, are the moments of $\mathbf{A}_a$ about the coördinate axes, and (X_1, Y_1, Z_1) are the resolved parts of $\mathbf{A}$ along these axes. In order to obtain a geometric interpretation of the formula (17.9), we may suppose that a and b are the ends of the common perpendicular to the axes of $\mathbf{A}_a$ and $\mathbf{B}_b$. Denoting by θ the angle from $\mathbf{A}$ to $\mathbf{B}$ measured positively around $\mathbf{r}_{ab}$ (and lying in the interval $-180° \leqq \theta < 180°$), we have at once an expression for M_{12} in terms of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$, the shortest distance δ between the axes of $\mathbf{A}_a$ and $\mathbf{B}_b$, and θ. In fact, $M_{12} = (\mathbf{A}.\mathbf{r}_{ab}.\mathbf{B}) = (\mathbf{r}_{ab}.[\mathbf{B}.\mathbf{A}])$. The magnitude of $[\mathbf{B}.\mathbf{A}]$ is $\pm AB \sin\theta$, and its direction is the same as or opposite to that of $\mathbf{r}_{ab}$ according as θ is negative or positive. Since the angle between $\mathbf{r}_{ab}$ and $[\mathbf{B}.\mathbf{A}]$ is 0° or $-180°$, the scalar product $(\mathbf{r}_{ab}.[\mathbf{B}.\mathbf{A}])$ is $-\delta AB \sin\theta$, where δ is the magnitude of $\mathbf{r}_{ab}$, that is, the length of the common perpendicular to the two axes. We have, therefore, the expression

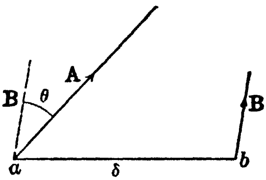

FIG. 9

$$M_{12} = -\delta AB \sin\theta. \tag{17.10}$$

This result gives us the expression $V = \frac{1}{6}\, ab\, \delta \sin\theta$ for the volume of a tetrahedron two of whose opposite edges have lengths a and b respectively, the shortest distance between these edges being δ, and θ being the angle between them; for (17.8) tells us that M_{12} is the volume of a parallelepiped which has representative segments of $\mathbf{A}$, $\mathbf{B}$, and $\overrightarrow{ab}$ as coterminous edges.

The two sliding segments are said to be positively related when M_{12} is positive, and negatively related when M_{12} is negative. We see, from (17.10), that the segments are positively or negatively related according as θ is negative or positive. In order that the relative moment M_{12} shall be zero when neither of the vectors $\mathbf{A}$ or $\mathbf{B}$ is zero, either δ or $\sin\theta$ must be zero. Thus, if $M_{12} = 0$, the axes of the two localized vectors $\mathbf{A}_a$ and $\mathbf{B}_b$ must either meet or be parallel. Both cases are included in the statement that the necessary and sufficient condition for the relative moment of two localized vectors to be zero is that their axes should be coplanar.

18. Systems of localized vectors. If we have a system of localized vectors, the vector moment of the system about any point is defined as the sum of the vector moments of the various localized vectors of the system about that point. Similarly, the moment of the system about any axis is defined as the sum of the moments of the various localized vectors of the system about that axis. Thus, if $(X_r, Y_r, Z_r, L_r, M_r, N_r)$ are the coördinates of the rth localized vector of the system, the moments of the system about the coördinate axes are

$$\sum_{r=1}^{n} L_r, \quad \sum_{r=1}^{n} M_r, \quad \sum_{r=1}^{n} N_r, \tag{18.1}$$

respectively, where there are supposed to be n localized vectors in the system. Denoting by $\mathbf{G}$ the vector which has these three quantities as components, $\mathbf{G}$ is the vector moment of the system about the origin o. Denoting by $\mathbf{R}$ the nonlocalized vector which has as components the three quantities

$$\sum_{r=1}^{n} X_r, \quad \sum_{r=1}^{n} Y_r, \quad \sum_{r=1}^{n} Z_r, \tag{18.2}$$

we see, from (17.4), that the vector moment of the system about any point a is given by the expression

$$\mathbf{G} - [\mathbf{r}_{oa}.\mathbf{R}]. \tag{18.3}$$

The six quantities

$$\begin{aligned} X &= \sum_{r=1}^{n} X_r, & Y &= \sum_{r=1}^{n} Y_r, & Z &= \sum_{r=1}^{n} Z_r, \\ L &= \sum_{r=1}^{n} L_r, & M &= \sum_{r=1}^{n} M_r, & N &= \sum_{r=1}^{n} N_r \end{aligned} \tag{18.4}$$

are known as the coördinates of the system.

It follows, from (18.3), that the vector moment of any system of localized vectors about any point in space is determined by the coördinates of the system. We shall say that two systems of localized vectors are equivalent when they have the same vector moment about any arbitrary point in space, and it follows that two systems are equivalent when they have the same coördinates. This sufficient condition for equivalence is readily seen to be necessary; for we have, from (18.3), on distinguishing the assumed equivalent system from the original one by primes,

$$\mathbf{G} - [\mathbf{r}_{oa}.\mathbf{R}] = \mathbf{G}' - [\mathbf{r}_{oa}.\mathbf{R}'],$$

or

$$\mathbf{G} - \mathbf{G}' = [\mathbf{r}_{oa}.(\mathbf{R} - \mathbf{R}')]. \tag{18.5}$$

Since $\mathbf{r}_{oa}$ is an arbitrary vector, and since $\mathbf{G}-\mathbf{G}'$ does not depend on the position of the point a, it follows, from (18.5), that

$$\mathbf{G}=\mathbf{G}' \quad \text{and} \quad \mathbf{R}=\mathbf{R}'; \tag{18.6}$$

that is, the two systems have the same coördinates.

19. The invariants of a system of localized vectors. The magnitude of the vector $\mathbf{R}$ is the scalar R; and this quantity is invariant, in the sense that it is independent of the choice of the particular frame used to present the localized vectors. Similarly, the magnitude G of the vector $\mathbf{G}$ is the same for all reference frames having a common origin o. These quantities are, moreover, invariant in another sense. Their value is unaltered not only by a change of reference frame but also by a change from one system of localized vectors to an equivalent one. Instead of considering G, whose value depends on the position of the origin o of the reference frame, we may consider the scalar product $(\mathbf{R}.\mathbf{G})$. It is apparent, from (18.3), that this is independent of the position of the origin o; for if a change is made to a new origin a, $\mathbf{G}$ becomes $\mathbf{G}-[\mathbf{r}_{oa}.\mathbf{R}]$, and $(\mathbf{R}.\mathbf{G})$ is unaltered since $(\mathbf{R}.[\mathbf{r}_{oa}.\mathbf{R}])=0$.

We have, then, for any system of localized vectors, two invariants,

$$R \quad \text{and} \quad (\mathbf{R}.\mathbf{G}). \tag{19.1}$$

These are unaltered by a change of the reference frame or by a change from one equivalent system to any other. They are known, respectively, as the first and second invariants of the system. The second invariant will be denoted by the symbol H, so that $H=(\mathbf{R}.\mathbf{G})$. The first invariant is the magnitude of the sum of all the *nonlocalized* vectors which have as representative segments the representative segments of the localized vectors of the system.

We shall now try to find the simplest system which is equivalent to a given system. Let us first consider the case where R is different from zero. Then, if H is not zero, it is impossible to find a single segment equivalent to the given system. For in a system consisting of a single segment the invariant H is, from (17.5), zero. If H is zero and R is not zero, there is a single segment equivalent to the given system. It is, in fact, the segment whose six coördinates are the coördinates of the system. Its axis is given by the equations (17.6). We may call this sliding segment the "sum" of the representative sliding segments of the system, and the localized vector which it represents the "sum" of the localized vectors of the system. If, for example, we wish to find the "sum" of two localized vectors, we see, from the fact that

$$\mathbf{R} = (X_1 + X_2,\ Y_1 + Y_2,\ Z_1 + Z_2),$$

$$\mathbf{G} = (L_1 + L_2,\ M_1 + M_2,\ N_1 + N_2),$$

and

$$X_1L_1 + Y_1M_1 + Z_1N_1 = 0,$$

$$X_2L_2 + Y_2M_2 + Z_2N_2 = 0, \qquad \text{[By 17.5]}$$

that the necessary and sufficient condition $H = 0$ amounts to

$$M_{12} = X_1L_2 + Y_1M_2 + Z_1N_2 + X_2L_1 + Y_2M_1 + Z_2N_1 = 0.$$

Hence, by (17.10), the two sliding segments must be coplanar. Thus it is not possible to add two localized vectors, and have the sum a localized vector, unless their axes are coplanar. If they are coplanar and meet, the single sliding segment which is their sum passes through their point of intersection and is constructed by means of the parallelogram law. If they are parallel, the single equivalent segment is parallel to each of them in the plane determined by them. It divides any transversal of the original two segments inversely in the ratio of their lengths. We emphasize again the fact that this definition of the sum of two vectors proves nothing about the compounding of physical quantities. Thus the result of rotating a rigid body successively about two intersecting axes is not described by the localized vector which is the sum, as defined above, of the two coplanar localized vectors which describe, respectively, the two separate rotations.

It is always possible to find in an infinity of ways two localized vectors which, taken together, form a system equivalent to a given system (**R**, **G**). In fact, if ($\mathbf{R}_1$, $\mathbf{G}_1$) and ($\mathbf{R}_2$, $\mathbf{G}_2$) refer, respectively, to the two desired segments, we have

$$\mathbf{R}_2 = \mathbf{R} - \mathbf{R}_1, \quad \mathbf{G}_2 = \mathbf{G} - \mathbf{G}_1. \tag{19.2}$$

Since $(\mathbf{R}_2.\mathbf{G}_2) = 0$ and $(\mathbf{R}_1.\mathbf{G}_1) = 0$, this gives

$$(\mathbf{R}.\mathbf{G}) = H = (\mathbf{R}.\mathbf{G}_1) + (\mathbf{G}.\mathbf{R}_1). \tag{19.3}$$

The first segment may be taken arbitrarily, except that it must satisfy the equation (19.3). The second segment is then given by (19.2).

We may now consider the case where $R = 0$. It follows, from (18.3), that the system of localized vectors has the same vector moment $\mathbf{G}$ about all points in space. The second invariant, H, is also zero in this case. If $\mathbf{G} = \mathbf{0}$, the system has zero moment about all points in space and may be called the zero system. The simplest equivalent system consists of two equal and opposite localized vectors (that is, the axes of the localized vectors lie along the same straight line).

If $\mathbf{G} \neq \mathbf{0}$, a system, consisting of two segments, equivalent to the given system may be found as follows: Since $R = 0$, $\mathbf{R}_2 = -\mathbf{R}_1$; that is, the two segments must be equal and parallel and oppositely directed. From the fact that $H = 0$, and from (19.3), we see that each of these segments must be perpendicular to $\mathbf{G}$. In fact, the plane of the two segments must be perpendicular to $\mathbf{G}$; for the moment vector of the system (consisting of the two equal and oppositely directed parallel segments) about all points in space is the same, since $\mathbf{R}$ is equal to $\mathbf{0}$ for this system. The value of this constant moment is $[\mathbf{r}_{a_1a_2}.\mathbf{R}_2]$, where a_1 and a_2 are any two points, one on each of the two segments. Since this is equal to $\mathbf{G}$, we see that $\mathbf{G}$ is perpendicular not only to $\mathbf{R}_2$ but to $\mathbf{r}_{a_1a_2}$.

The axis of one of the segments may be chosen arbitrarily, so long as it is perpendicular to $\mathbf{G}$, and its length may be taken arbitrarily. The second sliding segment is then determined uniquely by the fact that it is equal and parallel and oppositely directed to the first, and satisfies the equation

$$\mathbf{G} = [\mathbf{r}_{a_1a_2}.\mathbf{R}_2]. \tag{19.4}$$

Such a pair of equal and oppositely directed parallel segments is called a couple, and the plane of the segments is called the plane of the couple. $\mathbf{G}$, as defined by (19.4), is called the moment of the couple, and the direction of $\mathbf{G}$ is called the direction of the couple.

We have, then, the result that any system of localized vectors for which R, and consequently H, is zero, is equivalent to an infinity of couples. Such a system is called, briefly, a couple. The moment $\mathbf{G}$ of the system, being the same for all points, is called the moment of the couple. Its direction is the direction of the couple, and any plane perpendicular to it is called a plane of the couple.

20. Reduction of a system of parallel localized vectors. Let $\mathbf{A}_0$ denote the unit nonlocalized vector whose representative segment is parallel to the various sliding segments that represent the localized vectors of the parallel system. If a_r is any point on the axis of the rth sliding segment, the coördinates of this segment are given by the presentation of the vectors

$$\mathbf{R}_r = m_r\mathbf{A}_o \quad \text{and} \quad \mathbf{G}_r = [\mathbf{r}_{oa_r}.m_r\mathbf{A}_o], \tag{20.1}$$

where m_r is a scalar quantity. The $\mathbf{R}$ and $\mathbf{G}$ of the system are accordingly given by

$$\mathbf{R} = (\Sigma_r m_r)\mathbf{A}_o, \quad \mathbf{G} = \Sigma_r[\mathbf{r}_{oa_r}.m_r\mathbf{A}_o] = \Sigma_r[m_r\mathbf{r}_{oa_r}.\mathbf{A}_o]. \tag{20.2}$$

The invariant H of the system is accordingly zero. There are two cases to be considered according as $R = 0$ and $R \neq 0$. If $R = 0$, we have $\Sigma_r m_r = 0$. The system is a couple whose moment vector $\mathbf{G}$ is perpendicular to the common direction of the axes of the various sliding segments of the system.

If $R \neq 0$, we write $\Sigma_r m_r = m$, $\Sigma_r(m_r\mathbf{r}_{oa_r}) = m\mathbf{r}_{oa}$; and we see that the system is equivalent to a single segment whose coördinates are given by the presentation of the two vectors $\mathbf{R} = m\mathbf{A}_o$, $\mathbf{G} = [\mathbf{r}_{oa}.m\mathbf{A}_o]$. This segment is parallel to the common direction of the various sliding segments of the system and passes through the point a, satisfying the equation

$$\Sigma_r(m_r\mathbf{r}_{oa_r}) = m\mathbf{r}_{oa}. \tag{20.3}$$

The point a, defined by this equation, is the mean center of the points a_r, where a_r is given the weighting factor m_r.

21. The central axis of a system of localized vectors. When neither of the invariants R and H of a system of localized vectors is equal to zero, it is possible to find in an infinity of ways a system, consisting of the combination of a single segment and a couple, equivalent to the given system. This *reduction* of the system is usually more convenient than the reduction to a system of two sliding segments mentioned in § 19. If $(\mathbf{R}_1, \mathbf{G}_1)$ refer to the segment, and $\mathbf{G}_2$ to the couple, we must have

$$\mathbf{R} = \mathbf{R}_1, \quad \mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2. \tag{21.1}$$

From these equations, and the fact that $(\mathbf{R}_1.\mathbf{G}_1) = 0$, by (17.5), we derive

$$H = (\mathbf{R}.\mathbf{G}) = (\mathbf{R}.\mathbf{G}_2). \tag{21.2}$$

The moment $\mathbf{G}_2$ of the couple may be chosen arbitrarily so long as it satisfies this equation. It has an arbitrary direction, save for the fact that it must not be perpendicular to $\mathbf{R}$; and its magnitude is determined, once its direction is chosen, by the equation (21.2). The coördinates of the segment then follow from (21.1).

A particularly simple and important equivalent system consisting of a segment and a couple is obtained when the direction of the couple is the same as, or opposite to, the direction of the segment. In this case we have

$$\mathbf{G}_2 = h\mathbf{R}, \tag{21.3}$$

where h is a scalar quantity. We see, from (21.2), that $H = hR^2$; so that

$$h = \frac{H}{R^2} \tag{21.4}$$

and is therefore determined by the invariants of the system. On using (21.1) and (21.3) we have

$$\mathbf{G}_1 = \mathbf{G} - \mathbf{G}_2 = \mathbf{G} - h\mathbf{R}. \tag{21.5}$$

This, combined with $\mathbf{R}_1 = \mathbf{R}$, determines the axis of the segment by means of the equations (17.6).

This combination of a sliding segment and a couple whose moment vector is parallel to the segment is called a wrench or a screw. The axis of the sliding segment is known as the axis of the screw or the *central axis* of the system of localized vectors. The quantity h of (21.3) and (21.4) is called the pitch of the wrench or screw.

22. The relative moment of two systems of localized vectors. If we have two systems of localized vectors whose coördinates are given, respectively, by the presentations of the two pairs of vectors $(\mathbf{R}_1, \mathbf{G}_1)$ and $(\mathbf{R}_2, \mathbf{G}_2)$, it follows, from (18.3), that the expression

$$H_{12} = (\mathbf{R}_1.\mathbf{G}_2) + (\mathbf{R}_2.\mathbf{G}_1) \tag{22.1}$$

is independent of the reference frame. For if a change to a new origin a is made, $\mathbf{G}_2$ becomes $\mathbf{G}_2 - [\mathbf{r}_{oa}.\mathbf{R}_2]$, and so $(\mathbf{R}_1.\mathbf{G}_2)$ becomes $(\mathbf{R}_1.\mathbf{G}_2) - (\mathbf{R}_1.\mathbf{r}_{oa}.\mathbf{R}_2)$. Similarly, $(\mathbf{R}_2.\mathbf{G}_1)$ becomes $(\mathbf{R}_2.\mathbf{G}_1) - (\mathbf{R}_2.\mathbf{r}_{oa}.\mathbf{R}_1)$, showing that H_{12} is unaltered, since $(\mathbf{R}_1.\mathbf{r}_{oa}.\mathbf{R}_2) + (\mathbf{R}_2.\mathbf{r}_{oa}.\mathbf{R}_1) = 0$, by (3.11).

The scalar quantity H_{12} is not only unchanged when a change of the reference frame is made, but its value is unaltered, as follows from its definition and (18.6), when either of the two systems is replaced by an equivalent system. It is called the relative moment or the mutual invariant of the two systems. When it has the value zero, the two systems are said to be reciprocal to each other or to be in involution.

Let us suppose each system reduced to its equivalent screw. The coördinates of the segment of the first screw are given by the presentations of the vectors $(\mathbf{R}_1, \mathbf{G}_1^*)$, where $\mathbf{G}_1^* = \mathbf{G}_1 - h_1\mathbf{R}_1$, h_1 being the pitch of the first system (see (21.5)). We may, then, replace $\mathbf{G}_1$ by $\mathbf{G}_1^* + h_1\mathbf{R}_1$ in (22.1); and, doing the same thing for the second system, we find

$$\begin{aligned} H_{12} &= (\mathbf{R}_1.\mathbf{G}_2^* + h_2\mathbf{R}_2) + (\mathbf{R}_2.\mathbf{G}_1^* + h_1\mathbf{R}_1) \\ &= (\mathbf{R}_1.\mathbf{G}_2^*) + (\mathbf{R}_2.\mathbf{G}_1^*) + (h_1 + h_2)(\mathbf{R}_1.\mathbf{R}_2). \end{aligned}$$

The expression $(\mathbf{R}_1.\mathbf{G}_2^*) + (\mathbf{R}_2.\mathbf{G}_1^*)$ is, by (17.9), the relative moment of the two segments, and this is, by (17.10), equal to $-\delta R_1R_2 \sin\theta$, where δ is the shortest distance between the axes of the two sliding segments and where θ is the angle between these two axes measured positively *from* the first *to* the second in the direction of the shortest distance from the first to the second. We may write $(\mathbf{R}_1.\mathbf{R}_2) = R_1R_2\cos\theta$, and we then obtain the result

$$H_{12} = R_1R_2[(h_1 + h_2)\cos\theta - \delta\sin\theta]. \tag{22.2}$$

a. The screw product of two screws. The relative moment $(\mathbf{R}_1.\mathbf{G}_2) + (\mathbf{R}_2.\mathbf{G}_1)$ of two systems of localized vectors may be called the scalar product of the two screws. It is often convenient to introduce a screw which is derived from the two given screws and which may be called their screw product. Thus we may consider, in any given frame, the two vectors $[\mathbf{R}_1.\mathbf{R}_2]$ and $[\mathbf{R}_1.\mathbf{G}_2] + [\mathbf{G}_1.\mathbf{R}_2]$. These may be taken as the first and second vectors $(\mathbf{R}, \mathbf{G})$ of a localized vector system which is called the screw product of the first and second screws. To see this, let a change of reference frame be made, the new origin being a, and denote by $\mathbf{r}$ the nonlocalized vector which has $\overrightarrow{oa}$ as a representative segment. Then the new presentations of the two screws are, respectively,

$$(\mathbf{R}_1, \mathbf{G}_1 + [\mathbf{R}_1.\mathbf{r}]) \quad \text{and} \quad (\mathbf{R}_2, \mathbf{G}_2 + [\mathbf{R}_2.\mathbf{r}]).$$

The new presentation of the screw product is therefore

$$\{[\mathbf{R}_1.\mathbf{R}_2], [\mathbf{R}_1.\mathbf{G}_2] + [\mathbf{G}_1.\mathbf{R}_2] + [\mathbf{R}_1.[\mathbf{R}_2.\mathbf{r}]] + [[\mathbf{R}_1.\mathbf{r}].\mathbf{R}_2]\},$$

or

$$(\mathbf{R}, \mathbf{G} + [\mathbf{R}.\mathbf{r}]), \tag{22.3}$$

on making use of the identity

$$[\mathbf{R}_1.[\mathbf{R}_2.\mathbf{r}]] + [\mathbf{R}_2.[\mathbf{r}.\mathbf{R}_1]] + [\mathbf{r}.[\mathbf{R}_1.\mathbf{R}_2]] = 0. \quad \text{[See (3.13}a\text{)}$$

The result (22.3) shows that the vectors

$$\mathbf{R} = [\mathbf{R}_1.\mathbf{R}_2] \quad \text{and} \quad \mathbf{G} = [\mathbf{R}_1.\mathbf{G}_2] + [\mathbf{G}_1.\mathbf{R}_2]$$

may be used as the first and second vectors of a localized vector system. It is at once apparent that the screw product of two screws, like the vector product of two vectors, is not commutative. An interchange in the order of the factor screws changes the sign of the product.

If we introduce the central axes of the two screws and take any two points o_1 and o_2 on the first and second central axes respectively, the two screws are presented in an arbitrary reference frame, with origin at any point o, by the two pairs of vectors

$$(\mathbf{R}_1, h_1\mathbf{R}_1 + [\mathbf{R}_1.\mathbf{r}_1]) \quad \text{and} \quad (\mathbf{R}_2, h_2\mathbf{R}_2 + [\mathbf{R}_2.\mathbf{r}_2])$$

respectively, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are, respectively, the two nonlocalized vectors which have $\overrightarrow{o_1o}$ and $\overrightarrow{o_2o}$ as representative segments. The screw product is presented, then, in the frame whose origin is at o, by the vectors

$$[\mathbf{R}_1.\mathbf{R}_2] \quad \text{and} \quad [\mathbf{R}_1.\mathbf{R}_2]\,(h_1 + h_2) + [\mathbf{R}_1.[\mathbf{R}_2.\mathbf{r}_2]] + [[\mathbf{R}_1.\mathbf{r}_1].\mathbf{R}_2].$$

The second vector may be expanded to

$$(h_1 + h_2)\,[\mathbf{R}_1.\mathbf{R}_2] + (\mathbf{R}_1.\mathbf{r}_2)\mathbf{R}_2 - (\mathbf{R}_1.\mathbf{R}_2)\mathbf{r}_2 + (\mathbf{R}_1.\mathbf{R}_2)\mathbf{r}_1 - (\mathbf{r}_1.\mathbf{R}_2)\mathbf{R}_1.$$

By taking o_1 and o_2 as the ends of the common perpendicular to the two central axes, and o any point on this common perpendicular, this simplifies to

$$(h_1 + h_2)\,[\mathbf{R}_1.\mathbf{R}_2] + (\mathbf{R}_1.\mathbf{R}_2)\mathbf{D}_{12}, \tag{22.4}$$

where $\mathbf{D}_{12}$ is the nonlocalized vector which has the common perpendicular *from* the first *to* the second central axis as a representative segment. It follows, from (22.4), that the central axis of the product of two screws is the common perpendicular to the central axes of the two factor screws; for both the vectors $[\mathbf{R}_1.\mathbf{R}_2]$ and $(h_1 + h_2)\,[\mathbf{R}_1.\mathbf{R}_2] + (\mathbf{R}_1.\mathbf{R}_2)\mathbf{D}_{12}$ lie along this common perpendicular. Denoting, as before, by θ the angle from the first central axis to the second, measured positively around the direction on the common perpendicular *from* the first axis *to* the second, the pitch of the product screw is

$$h_1 + h_2 + \delta \cot \theta, \tag{22.5}$$

where δ is the length of the common perpendicular.

A special case presents the screw product of two simple segments as a screw whose central axis is the common perpendicular to the two segments, its direction being from the first segment to the second, and the pitch of the product screw being $\delta \cot \theta$. Thus the screw product is unaltered when the two simple segments are rotated as an ordinary screw about their common perpendicular, the angle between the segments and their distance apart being maintained constant. We may, conversely, regard any localized vector system as the product of two segments of arbitrary lengths meeting the central axis of the system at right angles. The axis of the first of the segments, so long as it meets the central axis at right angles, is arbitrary, but, once it is chosen, the axis of the other segment is completely determined.

These results may be arrived at very concisely by use of Clifford's* complex unit ϵ, which is subject to the ordinary rules of algebra but is such that $\epsilon^2 = 0$. A localized vector system or screw is written in the form $\mathbf{R} + \epsilon\mathbf{G}$ in this notation, and is known as a bivector. The scalar product of two bivectors $\mathbf{R}_1 + \epsilon\mathbf{G}_1$ and $\mathbf{R}_2 + \epsilon\mathbf{G}_2$ is

$$(\mathbf{R}_1 . \mathbf{R}_2) + \epsilon[(\mathbf{R}_1 . \mathbf{G}_2) + (\mathbf{R}_2 . \mathbf{G}_1)]$$

and leads at once to the relative moment of two localized vector systems. The vector product of the two bivectors is

$$[\mathbf{R}_1 . \mathbf{R}_2] + \epsilon([\mathbf{R}_1 . \mathbf{G}_2] + [\mathbf{G}_1 . \mathbf{R}_2])$$

and is what we have called the screw product of the two screws.

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This is encyclopedic in character.

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* W. K. Clifford, "Preliminary Sketch of Biquaternions," Proceedings of the London Mathematical Society, Vol. 4 (1871–1873), pp. 381–395. Reprinted in Clifford's "Mathematical Papers" (The Macmillan Company, 1882), pp. 181–200.

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EXERCISES*

1. If $u = f(r_1, r_2)$, where r_1 and r_2 are the distances of a variable point P from two fixed points P_1 and P_2, show that

$$\operatorname{grad} u = \frac{\partial f}{\partial r_1} \operatorname{grad} r_1 + \frac{\partial f}{\partial r_2} \operatorname{grad} r_2.$$

Hence show how to construct the normal to the level surface $u =$ constant which passes through any given point P. Treat, in particular, the cases $u = r_1 + r_2$ (level surfaces, ellipsoids), $u = r_1 - r_2$ (level surfaces, hyperboloids), $u = r_1 r_2$ (level surfaces, surfaces of revolution whose meridian curves are Cassinian ovals).

2. Extend the construction of Exercise 1 to take care of the level surfaces of the function $u = f(r_1, r_2, \cdots, r_n)$, where $(r_1, \cdots, r_n)$ are the distances of a variable point P from n fixed points $P_1, P_2, \cdots, P_n$.

3. Show how to find the points in space where the function $u = f(r_1, r_2, r_3)$ has a stationary value. (Here **grad** u must be **0**, so that the construction for the normal must break down.) Apply in particular to the function $u = r_1 + r_2 + r_3$.

4. If the variable point in Exercise 3 travels along a curve, show how to find the points on this curve where u has a stationary value. (Here **grad** u must be normal to the curve).

5. Repeat Exercise 4 assuming that the variable point moves along a given surface.

6. Show that the point for which the function $u = r_1^2 + r_2^2 + \cdots + r_n^2$ has a minimum value is the mean center of the points $P_1, \cdots, P_n$ from which the distances $(r_1, \cdots, r_n)$ are measured.

7. Show that div $[\mathbf{A}.\mathbf{B}] = (\mathbf{B}.\text{curl } \mathbf{A}) - (\mathbf{A}.\text{curl } \mathbf{B})$.

8. If **A** and **B** are any two vector fields, show that the quantities $A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}$ etc., (the other two expressions being obtained by writing B_y and B_z, in turn, for B_x in this expression) are the components of a vector. This vector is usually denoted by the symbol $(\mathbf{A}.\nabla)\mathbf{B}$.

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*Most of the exercises on this and the following chapters have been selected from the standard texts mentioned on page 103.

9. Express curl $[\mathbf{A}.\mathbf{B}]$ and grad $(\mathbf{A}.\mathbf{B})$ in terms of the vector of Exercise 8.

10. Prove that the integral $\int(\mathbf{r}.\mathbf{v})\,dS$ taken around any closed surface S is three times the volume inclosed by S.

11. State the theorem corresponding to that of Exercise 10 when the closed surface is replaced by a closed plane curve.

12. Calculate the position of the central axis and the pitch of the wrench equivalent to the system of three forces (0, 15, 0), (10, 0, 0), (0, 0, 21) whose representative segments pass, respectively, through the points (0, 0, 10), (0, 0, 0), and (5, 0, 0).

13. A circular plate is supported horizontally at its center. Show how to arrange three given weights around its circumference so as to preserve the equilibrium.

14. A balance has slightly unequal arms when the indicator points to zero. When a body weighs p grams in one pan and q grams in the other, what is its true weight?

15. If four forces are in equilibrium, prove that the invariant of any two is equal to that of the remaining two; also that the invariant of any three is zero.

16. Forces act at the corners of a tetrahedron perpendicularly to the opposite faces and proportionally to their areas. Prove that they are in equilibrium if they act either all inwards or all outwards.

17. Twelve equal forces occupy the edges of a cube, the parallel forces acting in the same direction. Prove that their central axis is a diagonal of the cube. If the forces are replaced by twelve equal couples whose axes occupy the edges, show that their central axis is parallel to a diagonal.

18. Any number of forces are represented in magnitude and direction by straight lines $\overrightarrow{A_1A'_1}$, $\overrightarrow{A_2A'_2}$, $\cdots$, $\overrightarrow{A_nA'_n}$, and G and G' are the centroids of the points $(A_1, A_2, \cdots, A_n)$ and $(A'_1, A'_2, \cdots, A'_n)$ respectively. Show that these forces, transferred parallel to themselves to act at a point, have a resultant which is represented in magnitude and direction by $\overrightarrow{nGG'}$.

19. If the forces in Exercise 18 are localized, show that the central axis is parallel to $\overrightarrow{GG'}$. If the representative segments intersect any plane perpendicular to $\overrightarrow{GG'}$ in the points $(B_1, B_2, \cdots, B_n)$, prove that the central axis intersects this plane in the center of gravity of particles placed at $(B_1, B_2, \cdots, B_n)$ whose weights are proportional to the resolved parts of the forces parallel to $\overrightarrow{GG'}$.

20. Forces (la, mb, nc) act in three nonintersecting edges of a parallelepiped, where (a, b, c) are the lengths of these edges. Prove that the product of the force and couple of the equivalent wrench is $(lm + mn + nl)V$, where V is the volume of the parallelepiped.

21. If a system consisting of two forces whose lines of action are given and a couple whose plane is given admit of a single resultant, prove that the direction of this resultant lies upon a certain hyperbolic paraboloid.

22. Show that a system of coplanar forces can be reduced to three forces which act along the sides of any triangle taken arbitrarily in the plane. Show also how to find these forces.

23. Find the components of a given vector **A** parallel to a given direction and a given plane.

HINT. If $\mathbf{n}$ is the unit vector along the given direction, and $\mathbf{p}$ is a unit vector normal to the given plane, $\mathbf{A} = \alpha\mathbf{n} + \beta\mathbf{r}$, where $\mathbf{r}$ is a vector perpendicular to both $\mathbf{p}$ and $[\mathbf{A}.\mathbf{n}]$. Writing $\mathbf{r} = [[\mathbf{A}.\mathbf{n}].\mathbf{p}]$, we find $\alpha = \frac{(\mathbf{A}.\mathbf{p})}{(\mathbf{n}.\mathbf{p})}$ and $\beta = -\frac{1}{(\mathbf{n}.\mathbf{p})}$, giving the result

$$\mathbf{A} = \frac{(\mathbf{A}.\mathbf{p})}{(\mathbf{n}.\mathbf{p})}\mathbf{n} + \frac{[\mathbf{p}.[\mathbf{A}.\mathbf{n}]]}{(\mathbf{n}.\mathbf{p})}.$$

24. Show that any system of localized vectors can be reduced to six localized vectors whose axes lie along the edges of a given tetrahedron.

25. Five localized vectors are equivalent to a zero system; show that their axes intersect two straight lines which may be coincident or imaginary. (Möbius.)

26. Prove that any system of parallel localized vectors is equivalent to a system of three parallel localized vectors whose axes pass through three given points.

27. The end point of a vector field $\mathbf{A}(t)$ traces out a curve as t varies (the initial point being supposed fixed); the curve traced out by the end point of the derivative $\dot{\mathbf{A}}(t) = \frac{d\mathbf{A}}{dt}$ is called the hodograph of the vector field (the initial point being again supposed fixed). Find the hodograph of a vector field whose end point traces out an ellipse around its initial point as focus in such a way that the rate at which areas are swept out by the radius vector is constant.

CHAPTER II

THE DISPLACEMENT AND MOTION OF A RIGID BODY

23. The rigid body with one fixed point. The concept of a rigid body as one having the property that the distance between any two of its points does not alter is inherent in the postulates of Euclidean geometry, — the congruence postulate, for example (see § 1). It is closely allied with the conception of time, and any difficulties arising in connection with the time idea will also be present in connection with the idea of rigidity. Some of these difficulties will be considered later.

In any displacement of a rigid body, lengths, areas, and volumes are unaltered. Such a displacement we shall call, for brevity, a rigid displacement. Let us suppose that three points of the rigid body which do not lie in the same straight line remain fixed in position. Then any fourth point M will also remain fixed, and no point can actually change its position. For, since the distances r_1, r_2, r_3 of M from the three fixed points P_1, P_2, P_3 are given, M must be one of the two points of intersection of the three spheres whose centers are at P_1, P_2, and P_3 and whose radii are r_1, r_2, and r_3 respectively. These two points of intersection, M_1 and M_2, are symmetrically situated with respect to the plane $P_1P_2P_3$, and each may be said to be the reflection of the other in this plane. The volume of the parallelepiped whose coterminous edges are M_1P_1, M_1P_2, and M_1P_3 is equal to the volume of the parallelepiped whose coterminous edges are M_2P_1, M_2P_2, and M_2P_3. If, however, we denote the nonlocalized vectors which have $\overrightarrow{MP_1}$, $\overrightarrow{MP_2}$, and $\overrightarrow{MP_3}$ as representative segments by $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ respectively, the scalar triple product $(\mathbf{r}_1.\mathbf{r}_2.\mathbf{r}_3)$ (whose numeric value is the volume of the parallelepiped which has MP_1, MP_2, and MP_3 as coterminous edges) will have the same absolute value but opposite signs at M_1 and M_2. Since not only the absolute value but also the sign of this scalar triple product is conserved in a rigid displacement, the position of M is unique, and there can be no actual displacement of the rigid body. The mathematical operation by which every point M_1 of a rigid body is transformed into its image M_2 in any plane does not affect the distance between any two points of the rigid body, but it is nevertheless not a possible displacement of the rigid body. We shall call such an operation reflection of the body in the plane.

There are, then, two operations, changing the positions of a set of material points in space, which have the property that the distance between any two points of the set is the same after the operation as it was before it. These two operations are distinguished from one another by the fact that one of them, the rigid displacement, leaves the scalar triple product $(\mathbf{r}_1.\mathbf{r}_2.\mathbf{r}_3)$ unaltered, whereas the other, the planar reflection, changes the sign of this scalar triple product without altering its numeric value. It follows that the succession of an even number of planar reflections is a rigid displacement; and we find that the most general operation which changes the positions of a set of material points in space without altering the distance between any two points of the set is either a rigid displacement or a rigid displacement followed by a planar reflection.

If two points, P_1 and P_2, of the body are fixed in a rigid displacement, every point of the straight line joining them is fixed. Any other point must move, if at all, on a circle whose center is on the line of fixed points and whose plane is perpendicular to this line. If M is displaced to M', the dihedral angle θ from the half-plane MP_1P_2 to the half-plane $M'P_1P_2$ is independent of the position of M; and we choose such a direction on the line of fixed points that θ is measured positively around it, according to the right-hand-screw convention. The displacement is then termed a rotation of amount θ about the directed line, which is known as the axis of rotation. The vector displacement of any point M of the rigid body may, then, be completely described by means of a segment of length θ on the axis of rotation. The position of the initial point of the representative segment on the axis of rotation is immaterial, so that we have here an example of a sliding segment.

The mathematical calculation of the vector displacement $\overrightarrow{MM'}$ of any point M of the rigid body in terms of the coördinates of the sliding segment may be best carried out somewhat as follows: Let M' be the new position of M, and let the plane through MM' perpendicular to the axis of rotation meet it in the point C. Take the origin O of a rectangular Cartesian reference frame on the axis of rotation, and denote by $\mathbf{r}$ and $\mathbf{r}'$ the nonlocalized vectors which have $\overrightarrow{OM}$ and $\overrightarrow{OM'}$, respectively, as representative segments. If $\boldsymbol{\omega}$ denotes the unit nonlocalized vector along the axis of rotation, we have $(\mathbf{r}'.\boldsymbol{\omega}) = (\mathbf{r}.\boldsymbol{\omega}) = OC$; and so the nonlocalized vector which has $\overrightarrow{CM'}$ as a representative segment is $\mathbf{r}' - (\mathbf{r}.\boldsymbol{\omega})\boldsymbol{\omega}$. Let us resolve this vector along and perpendicular to $\overrightarrow{CM}$. If the angle of rotation is denoted by θ, and the common length of $\overrightarrow{CM}$ and $\overrightarrow{CM'}$ by a, our vector has a component $a \cos \theta$ along $\overrightarrow{CM}$ and a component $a \sin \theta$ perpendicular to $\overrightarrow{CM}$ (the right angle being measured positively around the axis of rotation). Now the vector product $[\mathbf{r}.\boldsymbol{\omega}]$ has the magnitude a, and one of its representative segments lies in the plane CMM', at right angles to $\overrightarrow{CM}$, the right angle being measured negatively around the axis of rotation. Since the vector which has $\overrightarrow{CM}$ as a representative segment is $\mathbf{r} - (\mathbf{r}.\boldsymbol{\omega})\boldsymbol{\omega}$, we have

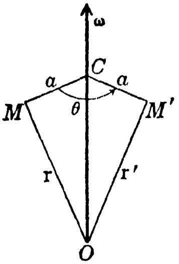

FIG. 10

$$\mathbf{r}' - (\mathbf{r}.\boldsymbol{\omega})\boldsymbol{\omega} = \{\mathbf{r} - (\mathbf{r}.\boldsymbol{\omega})\boldsymbol{\omega}\} \cos \theta - [\mathbf{r}.\boldsymbol{\omega}] \sin \theta. \qquad (23.1)$$

Upon solving for $\mathbf{r}'$, this becomes

$$\mathbf{r}' = \cos \theta\, \mathbf{r} + (\mathbf{r}.\boldsymbol{\omega})(1 - \cos \theta)\boldsymbol{\omega} - \sin \theta [\mathbf{r}.\boldsymbol{\omega}]. \qquad (23.2)$$

The required vector $\overrightarrow{MM'}$, being equal to $\mathbf{r}' - \mathbf{r}$, is therefore furnished by the equation

$$\overrightarrow{MM'} = (\cos \theta - 1)\mathbf{r} + (\mathbf{r}.\boldsymbol{\omega})(1 - \cos \theta)\boldsymbol{\omega} - \sin \theta [\mathbf{r}.\boldsymbol{\omega}]. \qquad (23.3)$$

It is convenient to introduce the notation $\rho = \cos \frac{\theta}{2}$ and $\mathbf{v} = \sin \frac{\theta}{2} \boldsymbol{\omega}$ so that

$$\mathbf{r}' = (2\rho^2 - 1)\mathbf{r} + 2(\mathbf{r}.\mathbf{v})\mathbf{v} - 2\rho [\mathbf{r}.\mathbf{v}]. \qquad (23.4)$$

If (x, y, z), (x', y', z'), and (λ, μ, ν) are, respectively, the presentations of the vectors $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{v}$ in the frame $OXYZ$, then, since $\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1$, the equation (23.4) is seen to be equivalent to the equations

$$\begin{aligned} x' &= (\rho^2 + \lambda^2 - \mu^2 - \nu^2)x + 2(\lambda\mu - \nu\rho)y + 2(\lambda\nu + \mu\rho)z, \\ y' &= 2(\mu\lambda + \nu\rho)x + (\rho^2 + \mu^2 - \lambda^2 - \nu^2)y + 2(\mu\nu - \lambda\rho)z, \qquad (23.5) \\ z' &= 2(\nu\lambda - \mu\rho)x + 2(\nu\mu + \lambda\rho)y + (\rho^2 + \nu^2 - \lambda^2 - \mu^2)z. \end{aligned}$$

The equation (23.4) may be solved for $\mathbf{r}$ in terms of $\mathbf{r}'$ by merely changing the sign of $\mathbf{v}$, without altering that of ρ. For the rotation which sends the points M' into the points M differs from that which sends the points M into the points M' merely by a reversal of the axis of rotation. In this way we obtain the following equation giving $\mathbf{r}$ in terms of $\mathbf{r}'$:

$$\mathbf{r} = (2\,\rho^2 - 1)\mathbf{r}' + 2(\mathbf{r}'.\mathbf{v})\mathbf{v} + 2\,\rho[\mathbf{r}'.\mathbf{v}]. \tag{23.6}$$

Instead of considering the relation of *two* points M and M' to a *single* fixed reference frame $OXYZ$, we may now consider the relation of a single point M' with reference to a frame which rotates with the rigid body. The coördinates of M' with reference to the new frame $OX'Y'Z'$ are the same as the coördinates (x, y, z) of M with respect to $OXYZ$; for since M moves with the frame $OXYZ$ its new position M' bears the same relation to the new position $OX'Y'Z'$ of the frame as its old position M bore to the old position $OXYZ$ of the reference frame. As before, (x', y', z') denote the coördinates of the point M' under discussion relative to $OXYZ$, and so the vector equation (23.6) furnishes us with the following table of direction cosines. (See the deduction of (23.5) and refer to (2.3).)

	x	y	z
x'	$\rho^2 + \lambda^2 - \mu^2 - \nu^2$	$2(\lambda\mu + \nu\rho)$	$2(\lambda\nu - \mu\rho)$
y'	$2(\mu\lambda - \nu\rho)$	$\rho^2 + \mu^2 - \nu^2 - \lambda^2$	$2(\mu\nu + \lambda\rho)$
z'	$2(\nu\lambda + \mu\rho)$	$2(\nu\mu - \lambda\rho)$	$\rho^2 + \nu^2 - \lambda^2 - \mu^2$

(23.7)

In comparing the vector equation (23.6) with (2.3) the reader will be careful to note that (x', y', z'), being the presentation of $\mathbf{r}'$ in the frame $OXYZ$, plays the rôle of (A_x, A_y, A_z), whereas (x, y, z), being the presentation of the same vector in the frame $OX'Y'Z'$, plays the rôle of (A'_x, A'_y, A'_z). We shall see shortly that any reference frame $OXYZ$ can be sent into any other reference frame $OX'Y'Z'$ having the same origin O by a rigid rotation about an axis through O. The table (23.7) expresses, then, the nine direction cosines giving the relative orientation of any two frames in terms of four parameters $(\lambda, \mu, \nu, \rho)$ which are connected by the relation $\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1$. If (l, m, n) are the direction cosines of the axis of rotation, and if θ is the angle of the rotation sending the frame $OXYZ$ into the frame $OX'Y'Z'$, we have

$$\lambda = l \sin\frac{\theta}{2}, \quad \mu = m \sin\frac{\theta}{2}, \quad \nu = n \sin\frac{\theta}{2}, \quad \rho = \cos\frac{\theta}{2}. \qquad (23.8)$$

A second specification of the nine direction cosines which give the relative orientation of two frames $OXYZ$ and $OX'Y'Z'$ having a common origin O is due to Euler and will be found very useful. Denote by ON that half of the line of intersection of the "equatorial" planes OXY and $OX'Y'$, around which the smaller angle θ from OZ to OZ' is measured positively, and call it the line of nodes. The line of nodes is therefore perpendicular to the plane OZZ'. Denote by OT and OT' the half-lines in the equatorial planes OXY and $OX'Y'$, respectively, which are behind ON by a right angle, the sense of positive measurement of angles being around OZ and OZ' respectively. The lines OT and OT' are in the plane OZZ' and are called the transverse axes in the two equatorial planes. Then the angle from OX to OT is denoted by ϕ. It is really a dihedral angle measuring the inclination of the half-plane ZOZ' to the half-plane ZOX, the sense of positive rotation being around OZ. The angle from OT' to OX' is denoted by ψ. It also is a dihedral angle, measured from the half-plane ZOZ' to the half-plane $Z'OX'$ positively around OZ'. It may save misunderstanding, later on, if attention is directed explicitly to the difference in the definition of the two angles ϕ and ψ: the first is measured *from OX to OT*, whereas the second is measured *from OT' to OX'*. The nine direction cosines may be expressed in terms of the three angles θ, ϕ, and ψ, which are known as the Eulerian angles. It follows, from the definition of these angles, that the frame $OXYZ$ may be brought into coincidence with the frame $OX'Y'Z'$ by the following series of operations, performed in the order indicated:

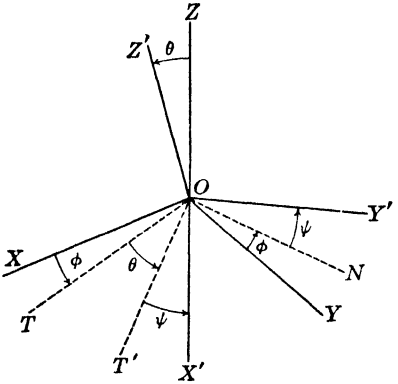

FIG. 11

1. A rotation through an angle ϕ about OZ.
2. A rotation through an angle θ about ON, the new position of OY.
3. A rotation through an angle ψ about OZ', the new position of OZ.

Similarly the $OX'Y'Z'$ frame may be brought into coincidence with the $OXYZ$ frame by the following series of operations, performed in the order indicated:

1. A rotation through an angle $-\psi$ about OZ'.
2. A rotation through an angle $-\theta$ about ON, the new position of OY'.
3. A rotation through an angle $-\phi$ about OZ, the new position of OZ'.

Hence the rôles of the two frames may be interchanged by replacing ϕ by $-\psi$, θ by $-\theta$, and ψ by $-\phi$.

In order to find the actual expressions of the direction cosines as functions of the angles ϕ, θ, and ψ, we shall express the unit vectors ($\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$) along the axes of the $OX'Y'Z'$ frame as linear combinations of the unit vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) along the axes of the $OXYZ$ frame. Thus $\mathbf{i}'$ may be expressed as the sum of a vector $\cos\psi$ along $\overrightarrow{OT'}$ and a vector $\sin\psi$ along $\overrightarrow{ON}$. But a unit vector along OT' may be expressed as the sum of a vector $-\sin\theta$ along $\overrightarrow{OZ}$ and a vector $\cos\theta$ along $\overrightarrow{OT}$. Finally, a unit vector along $\overrightarrow{ON} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$, and a unit vector along $\overrightarrow{OT} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$. Hence we have

$$\begin{aligned}\mathbf{i}' = {} & (\cos\theta\cos\psi\cos\phi - \sin\phi\sin\psi)\mathbf{i} \\ & + (\cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi)\mathbf{j} - \cos\psi\sin\theta\,\mathbf{k}.\end{aligned}$$

Similarly, or $\left(\text{since the angle from } OT' \text{ to } OY' \text{ is } \psi + \frac{\pi}{2}\right)$ by merely writing $\psi + \frac{\pi}{2}$ for ψ in this expression,

$$\begin{aligned}\mathbf{j}' = {} & (-\cos\theta\cos\phi\sin\psi - \sin\phi\cos\psi)\mathbf{i} \\ & + (-\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi)\mathbf{j} + \sin\psi\sin\theta\,\mathbf{k}.\end{aligned}$$

A unit vector along $\overrightarrow{OZ'}$ may be expressed as the sum of a vector $\cos\theta$ along $\overrightarrow{OZ}$ and a vector $\sin\theta$ along $\overrightarrow{OT}$. Hence

$$\mathbf{k}' = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}.$$

We have, then, the following table of direction cosines:

	OX	OY	$OZ.$
OX'	$\cos\theta\cos\phi\cos\psi$ $-\sin\phi\sin\psi$	$\cos\theta\sin\phi\cos\psi$ $+\cos\phi\sin\psi$	$-\sin\theta\cos\psi$
OY'	$-\cos\theta\cos\phi\sin\psi$ $-\sin\phi\cos\psi$	$-\cos\theta\sin\phi\sin\psi$ $+\cos\phi\cos\psi$	$\sin\theta\sin\psi$
OZ'	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$

(23.9)

As a check on the accuracy of this table we note that the rows and columns interchange when ϕ is replaced by $-\psi$, θ by $-\theta$, and ψ by $-\phi$. Just as the second row is obtained from the first on replacing ψ by $\psi+\frac{\pi}{2}$, the second column is obtained from the first by replacing $-\phi$ by $-\phi+\frac{\pi}{2}$, that is, ϕ by $\phi-\frac{\pi}{2}$.

24. The composition of rotations about coplanar axes. A rotation of amount θ about a given axis is readily seen to be equivalent to a succession of two planar reflections in two planes through the axis of rotation, the angle from the first plane to the second being $\frac{\theta}{2}$. The position of the first plane is immaterial, so long as it passes through the axis of rotation; but the order in which the two reflections are carried out is essential. Reversing the order is equivalent to changing the sign of θ, or, if we wish, to changing the direction of the vector along the axis of rotation. The only case in which an interchange of the order of the reflections is unimportant is when the angle $\theta=\pi$; that is, when the two planes are at right angles. This analysis of a rotation into a succession of two planar reflections enables us to compound rotations about two coplanar axes. Let us first suppose that the axes are not parallel, and let them meet in a point O. Denoting by A and B the points in which the axes pierce the unit sphere which has O for its center (OA and OB being supposed to give the directions of the two axes), we choose

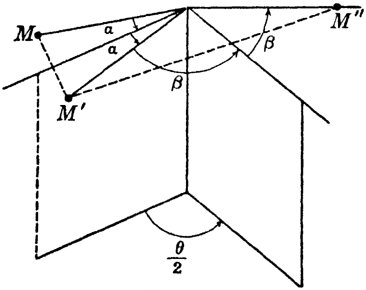

FIG. 12

the two planes in which the reflections equivalent to the rotation about OA are carried out, in such a way that the plane OAB is the second of the two planes. The two planes in which the reflections yielding the rotation about OB are carried out are then chosen so that OAB is the first of them. The two successive reflections in the plane OAB neutralize each other, and we see that the net result of the two rotations about OA and OB in succession is equivalent to a succession of two planar reflections, the first of the planes passing through OA, and the second through OB; that is, to a rotation about an axis OC passing through O. If we consider the spherical triangle ABC, the position of OC and the amount of the rotation about it are given by the following theorem:

A rotation about OA through twice the **internal** *angle at A, followed by a rotation about OB through twice the* **internal** *angle at B, is equivalent to a rotation about OC through twice the* **external** *angle at C.*

In the figure, OAC is the first plane in which the reflection is performed, and OAB is the second, so that the dihedral angle at A (that is, the angle A of the spherical triangle ABC) is half the angle θ_1 of the first rotation. Similarly, the angle B of the spherical triangle is half the angle θ_2 of the rotation about OB. Since the two reflections in the plane OAB neutralize each other, the final position M' of any point M may be obtained by first reflecting M in the plane OAC and then reflecting its new position in the plane OBC. We take the *external* angle of the spherical triangle at C, instead of the *internal* angle, as half the angle of the equivalent rotation about OC, since the angle of rotation is to be measured positively around OC according to the right-hand-screw convention.

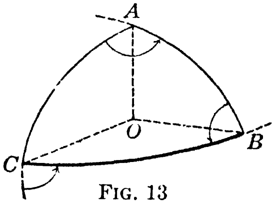

FIG. 13

This leads at once to a result, which is due to Euler, affecting the displacement of a rigid body which has one fixed point. Let any point M of the body be displaced to a position M'. Then, O being the fixed point, a rotation of proper amount about an axis through O perpendicular to the plane OMM' will send M into M'. Further displacement from the new position of the rigid body has O and M' as fixed points and is therefore a rotation about OM'. Thus the most general displacement of a rigid body having a fixed point is equivalent to a succession of two rotations about axes through the fixed point, and these are equivalent, as we have just seen, to a single rotation about an axis through the fixed point.

The spherical-triangle construction for the resultant of two rotations about concurrent axes enables us to determine the parameters $(\lambda, \mu, \nu, \rho)$ of the resultant rotation in terms of the parameters $(\lambda_1, \mu_1, \nu_1, \rho_1)$ and $(\lambda_2, \mu_2, \nu_2, \rho_2)$ of the separate rotations. Denoting the sides of the spherical triangle, as usual, by (a, b, c), and the angles by (A, B, C), we have

$$A = \frac{\theta_1}{2}, \quad B = \frac{\theta_2}{2}, \quad C = \pi - \frac{\theta}{2}, \quad \cos c = l_1 l_2 + m_1 m_2 + n_1 n_2. \tag{24.1}$$

The spherical-triangle formula

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \qquad \text{[See (3.22)]}$$

gives
$$\cos\frac{\theta}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos c,$$

or
$$\rho = \rho_1\rho_2 - \lambda_1\lambda_2 - \mu_1\mu_2 - \nu_1\nu_2. \tag{24.2}$$

In order to find (λ, μ, ν) we express the unit vector **C** along $\overrightarrow{OC}$ as a linear combination of the unit vectors **A** (along $\overrightarrow{OA}$) and **B** (along $\overrightarrow{OB}$), and the vector product [**A**.**B**] along, say, $\overrightarrow{OD}$. The formula (3.16) gives

$$(\mathbf{A}.\mathbf{B}.[\mathbf{A}.\mathbf{B}])\mathbf{C} = (\mathbf{C}.\mathbf{B}.[\mathbf{A}.\mathbf{B}])\mathbf{A} + ([\mathbf{A}.\mathbf{B}].\mathbf{A}.\mathbf{C})\mathbf{B} + (\mathbf{A}.\mathbf{B}.\mathbf{C})[\mathbf{A}.\mathbf{B}]. \tag{24.3}$$

Now
$$(\mathbf{A}.\mathbf{B}.[\mathbf{A}.\mathbf{B}]) = ([\mathbf{A}.\mathbf{B}].[\mathbf{A}.\mathbf{B}]) = \sin^2 c,$$

since the magnitude of [**A**.**B**] is $\sin c$, and

$$\begin{aligned}(\mathbf{C}.\mathbf{B}.[\mathbf{A}.\mathbf{B}]) &= ([\mathbf{C}.\mathbf{B}].[\mathbf{A}.\mathbf{B}])\\ &= (\mathbf{C}.\mathbf{A})(\mathbf{B}.\mathbf{B}) - (\mathbf{C}.\mathbf{B})(\mathbf{A}.\mathbf{B}) \qquad \text{[By (3.19)]}\\ &= \cos b - \cos a \cos c\\ &\qquad \text{[A, B, and C being unit vectors]}\\ &= \sin a \sin c \cos B. \qquad \text{[By (3.21)]}\end{aligned}$$

Similarly, $([\mathbf{A}.\mathbf{B}].\mathbf{A}.\mathbf{C}) = \sin b \sin c \cos A.$

Furthermore,

$$(\mathbf{A}.\mathbf{B}.\mathbf{C}) = ([\mathbf{A}.\mathbf{B}].\mathbf{C}) = \sin c \cos\left(\frac{\pi}{2} + h_c\right) = -\sin c \sin h_c,$$

where h_c is the altitude on side c of the spherical triangle.

Now $\sin h_c$ may be replaced by $\sin B \sin a$; and we have, from (24.3),

$$\mathbf{C} = \frac{\sin a}{\sin c}\cos B\,\mathbf{A} + \frac{\sin b}{\sin c}\cos A\,\mathbf{B} - \frac{\sin a}{\sin c}\sin B\,[\mathbf{A}.\mathbf{B}] \tag{24.4}$$

$$= \frac{\sin A}{\sin C}\cos B\,\mathbf{A} + \frac{\sin B}{\sin C}\cos A\,\mathbf{B} - \frac{\sin A}{\sin C}\sin B\,[\mathbf{A}.\mathbf{B}]. \tag{24.5}$$

Upon using the results of (24.1), this may be written

$$\sin\frac{\theta}{2}\mathbf{C} = \cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\mathbf{A} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathbf{B} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}[\mathbf{A}.\mathbf{B}], \tag{24.6}$$

or, equivalently,

$$\begin{aligned} \lambda &= \rho_2\lambda_1 + \rho_1\lambda_2 - \mu_1\nu_2 + \mu_2\nu_1, \\ \mu &= \rho_2\mu_1 + \rho_1\mu_2 - \nu_1\lambda_2 + \nu_2\lambda_1, \\ \nu &= \rho_2\nu_1 + \rho_1\nu_2 - \lambda_1\mu_2 + \lambda_2\mu_1. \end{aligned} \tag{24.7}$$

These relations, combined with (24.2), are the formulas required. They may be represented compactly by the following device. Let us call the rotation which is specified by $(\lambda, \mu, \nu, \rho)$ a *versor* and denote it by the symbol

$$\rho + \lambda i + \mu j + \nu k, \tag{24.8}$$

where (i, j, k) are complex quantities supposed to be subject to the distributive laws of addition and multiplication. The result of two versors

$$\rho_1 + \lambda_1 i + \mu_1 j + \nu_1 k \quad \text{and} \quad \rho_2 + \lambda_2 i + \mu_2 j + \nu_2 k$$

carried out in succession in this order is given by the product

$$(\rho_2 + \lambda_2 i + \mu_2 j + \nu_2 k)(\rho_1 + \lambda_1 i + \mu_1 j + \nu_1 k)$$

the complex quantities (i, j, k) being supposed to obey the following multiplication table:

$$\begin{aligned} i^2 = j^2 = k^2 = -1, \quad & ij = -ji = +k, \\ jk = -kj = +i, \quad & ki = -ik = +j. \end{aligned} \tag{24.9}$$

The versor (24.8) is a unit quaternion. It is obvious, from (23.4), that if each of the four quantities $(\lambda, \mu, \nu, \rho)$ is multiplied by the same number $k^{\frac{1}{2}}$, we have a rotation followed by an extension from the origin, that is, a rotation from (x, y, z) to (x', y', z') followed by a transformation of the type $x'' = kx'$, $y'' = ky'$, $z'' = kz'$. The four arbitrary quantities $(\lambda, \mu, \nu, \rho)$ are, then, said to build up, by means of an expression of the type (24.8), a quaternion; and $k = (\lambda^2 + \mu^2 + \nu^2 + \rho^2)^{\frac{1}{2}}$ is called the tensor of the quaternion.

a. The composition of rotations about parallel axes. It is evident that a succession of two planar reflections in parallel planes carries any point M to a point M', such that $\overrightarrow{MM'}$ is perpendicular to each of the planes, its direction being *from* the first *to* the second, and its length being equal to twice the distance between the parallel planes. This type of rigid displacement, in which all points have equal vector displacements, is known as a translation. If, now, we wish to compound two successive rotations about two parallel axes, we draw any plane perpendicular to the axes and let these axes pierce this plane in the points A and B respectively. The succession of the two rotations is equivalent to a single rotation about a parallel axis through a point C determined as in the spherical-triangle construction, the only difference being that the spherical triangle is replaced by a plane triangle ABC in the plane perpendicular to the axes of rotation. Since the external angle at C is equal to the sum of the internal angles at A and B, the amount θ of the single equivalent rotation is given by the formula

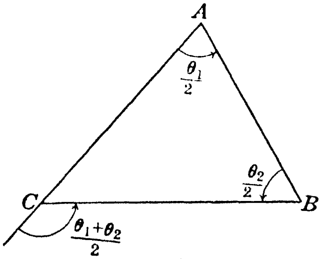

FIG. 14

$$\theta = \theta_1 + \theta_2. \tag{24.10}$$

When $\theta_1 + \theta_2 = 0$, the planes through AC and BC are parallel, and we have the result that the succession of two equal and opposite rotations about parallel axes is equivalent to a translation of amount $2\,d \sin \frac{\theta_1}{2}$, where d is the distance between the parallel axes and where θ_1 is the common magnitude of the two rotations. The direction of the translation is perpendicular to that plane which passes through the first axis and which must be turned through an angle $\frac{\theta_1}{2}$ in order to be brought into coincidence with the plane of the two axes. The translation is *from* the former plane *toward* the parallel plane through the second axis.

25. The general rigid displacement. If the vector displacements of all points of a rigid body are parallel to a given plane, the body may be said to suffer a plane displacement. If one point is fixed in such a displacement, all points in a line through this fixed point perpendicular to the given plane must be fixed (since any displacement of such a point parallel to the given plane would necessarily increase its distance from the fixed point), and we have rotation about an axis perpendicular to the given plane. If, in any plane displacement whatsoever, a point M moves into a position M', the displacement may be analyzed into the translation $\overrightarrow{MM'}$ followed by a rotation about an axis through M' perpendicular to the plane of displacement. The translation may be further analyzed into two equal and opposite rotations about axes which are perpendicular to the plane of displacement. Since the three rotations about parallel axes may be compounded into either a single rotation about a parallel axis or a translation perpendicular to the axes of rotation, we have the following result:

The most general plane rigid displacement is equivalent either to a translation parallel to the plane of displacement or to a rotation about an axis perpendicular to the plane of displacement.

We may now consider the most general displacement of a rigid body. If any point M is displaced to M', it follows, by Euler's theorem, that the displacement is equivalent to the translation $\overrightarrow{MM'}$ followed by a rotation about an axis $\overrightarrow{M'A}$ passing through M'. The translation $\overrightarrow{MM'}$ may be analyzed into a translation $\overrightarrow{MB}$ parallel to the axis of rotation $\overrightarrow{M'A}$ and a translation $\overrightarrow{BM'}$ perpendicular to $\overrightarrow{M'A}$. The displacement which consists of the translation $\overrightarrow{BM'}$ followed by the rotation about $\overrightarrow{M'A}$ is a plane displacement, the plane of displacement being perpendicular to $\overrightarrow{M'A}$. Applying the rule for plane displacements which has just been obtained, we obtain the following result:

The most general displacement of a rigid body is a translation, or can be analyzed into a translation followed by a rotation about an axis parallel to the direction of translation.

Such a displacement is called a screw, the axis of the rotation being known as the axis of the screw. Since any translation or rotation may be analyzed into two planar reflections, and since the order of reflection in two perpendicular planes may be interchanged, we may present a screw motion as follows:

Take, on the axis of the screw, two points, at a distance apart equal to one half the magnitude of the translation, such that the direction from the first to the second is that of the translation. Take any two lines, one through each of these points, perpendicular to the axis of the screw and such that the angle from the first to the second is one half the angle of rotation (the sense of positive measurement of angle being positively related to the axis of the screw). The screw is then equivalent to a rotation through two right angles about the first line, followed by a rotation through two right angles about the second line; for a rotation through two right angles about any line — that is, an inversion or reflection in the line — is equivalent to a succession of two planar reflections in any two perpendicular planes through the line.

We have, then, merely to choose as the *first* plane of the first rotation the plane containing the first line and the axis of the screw and as the *second* plane of the second rotation the plane containing the second line and the axis of the screw. Since the planes 1, 2, 3, and 4 of the four reflections are such that 1 is perpendicular to 2 and 3, the reflections may be changed from the order (1, 2, 3, 4) to the order (2, 3, 1, 4). The planes 2 and 3 being parallel, and perpendicular to the axis of the screw, the first two reflections, in the new order, give the translation while the last two give the rotation of the screw.

This analysis of a screw shows how to compound any two screws. We have merely to arrange that the second line for the first screw and the first line for the second screw shall each coincide with the common perpendicular to the axes of the two screws. The axis of the resultant screw is the common perpendicular to the first line for the first screw and to the second line for the second screw.

Since a rotation is a screw of zero pitch, this construction enables us to describe the screw which is equivalent to a succession of two rotations.

26. The time derivative of a vector with respect to a reference frame. If the components (A_x, A_y, A_z) of a vector **A** in any reference frame $OXYZ$ are functions of the time t, the three quantities $\left(\frac{dA_x}{dt}, \frac{dA_y}{dt}, \frac{dA_z}{dt}\right)$ present in the same frame a vector which we shall denote by $\frac{d\mathbf{A}}{dt}$. (See § 5.) This vector may be called the time derivative of the vector **A** with respect to the frame

OXYZ. It is essential to realize that the new vector derived in this way depends not only on the original vector **A** but on the frame *OXYZ* in which it was presented. We have already seen, in § 5, that the time derivatives of any vector with respect to frames which are at rest relatively to one another are the same. If, however, the frames are in motion relative to one another, the time derivatives will, in general, be different. Thus, let us consider a reference frame $P\xi\eta\zeta$ which we shall, for convenience, call fixed, and let us denote the time derivative of **A** with respect to this frame by $\dot{\mathbf{A}}$. This time derivative may be called the "absolute" time derivative of **A**, and the derivative with respect to the frame *OXYZ* the "relative" time derivative of **A**. In order to find the connection between the absolute and relative time derivatives, we write

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}, \tag{26.1}$$

where **i**, **j**, **k** denote unit vectors along the axes *OX*, *OY*, *OZ*, respectively, of the reference frame *OXYZ*. If the frame *OXYZ* is in motion with respect to the fixed frame $P\xi\eta\zeta$, the unit vectors **i**, **j**, **k** will have presentations in the fixed frame which will, in general, vary with the time. In forming, then, the absolute time derivative $\dot{\mathbf{A}}$ we shall have to regard **i**, **j**, and **k** as functions of *t*. Thus we obtain, from (26.1),

$$\dot{\mathbf{A}} = A_x\dot{\mathbf{i}} + \frac{dA_x}{dt}\mathbf{i} + A_y\dot{\mathbf{j}} + \frac{dA_y}{dt}\mathbf{j} + A_z\dot{\mathbf{k}} + \frac{dA_z}{dt}\mathbf{k}. \quad \text{[See § 5]} \tag{26.2}$$

We can now find the presentation of $\dot{\mathbf{A}}$ in the frame *OXYZ* by multiplying it scalarly by **i**, **j**, and **k** in succession. On account of the relations

$$(\mathbf{i}.\mathbf{i}) = 1, \quad (\mathbf{i}.\mathbf{j}) = 0, \quad (\mathbf{i}.\mathbf{k}) = 0, \quad \text{etc.}$$

we have, on differentiation with respect to *t*,

$$(\mathbf{i}.\dot{\mathbf{i}}) = 0, \quad (\mathbf{i}.\dot{\mathbf{j}}) + (\dot{\mathbf{i}}.\mathbf{j}) = 0, \quad (\mathbf{i}.\dot{\mathbf{k}}) + (\dot{\mathbf{i}}.\mathbf{k}) = 0, \quad \text{etc.} \tag{26.3}$$

Upon introducing the notation

$$\begin{gathered} \omega_x = (\mathbf{k}.\dot{\mathbf{j}}) = -(\mathbf{j}.\dot{\mathbf{k}}), \quad \omega_y = (\mathbf{i}.\dot{\mathbf{k}}) = -(\mathbf{k}.\dot{\mathbf{i}}), \\ \omega_z = (\mathbf{j}.\dot{\mathbf{i}}) = -(\mathbf{i}.\dot{\mathbf{j}}), \end{gathered} \tag{26.4}$$

we have

$$\begin{aligned} (\dot{\mathbf{A}})_x &= (\dot{\mathbf{A}}.\mathbf{i}) = \frac{dA_x}{dt} + \omega_y A_z - \omega_z A_y, \\ (\dot{\mathbf{A}})_y &= (\dot{\mathbf{A}}.\mathbf{j}) = \frac{dA_y}{dt} + \omega_z A_x - \omega_x A_z, \\ (\dot{\mathbf{A}})_z &= (\dot{\mathbf{A}}.\mathbf{k}) = \frac{dA_z}{dt} + \omega_x A_y - \omega_y A_x. \end{aligned} \tag{26.5}$$

The equations (26.5) show that the three expressions

$$(\omega_y A_z - \omega_z A_y, \quad \omega_z A_x - \omega_x A_z, \quad \omega_x A_y - \omega_y A_x)$$

present in the frame $OXYZ$ a vector, which is the vector difference $\dot{\mathbf{A}} - \frac{dA}{dt}$ between the absolute and relative time derivatives of the vector $\mathbf{A}$. Since this is true no matter what the vector $\mathbf{A}$ is, we know, from (3.18), that the three expressions $(\omega_x, \omega_y, \omega_z)$, defined by (26.4), present in the frame $OXYZ$ a vector. Denoting this vector by $\boldsymbol{\omega}$, we may rewrite the equations (26.5) in the vector form

$$\dot{\mathbf{A}} = \frac{d\mathbf{A}}{dt} + [\boldsymbol{\omega} . \mathbf{A}]. \tag{26.6}$$

As an example let us consider the special case where one of the unit vectors — for example, $\mathbf{k}$ — is constant with reference to the fixed, or absolute, frame. We may choose our axis $P\zeta$ of the fixed frame parallel to the axis OZ of the moving frame, and we may denote the dihedral angle from the half-plane $\zeta P\xi$ to the half-plane ZOX, measured positively around the common direction of the axes $P\zeta$ and OZ, by ϕ. Then the presentations, in the fixed frame $P\xi\eta\zeta$, of the vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are, respectively,

$$(\cos\phi, \sin\phi, 0), \quad (-\sin\phi, \cos\phi, 0), \quad (0, 0, 1).$$

Hence the presentations in this same frame of $(\dot{\mathbf{i}}, \dot{\mathbf{j}}, \dot{\mathbf{k}})$ are, by definition,

$$(-\sin\phi\,\dot{\phi}, \cos\phi\,\dot{\phi}, 0), \quad (-\cos\phi\,\dot{\phi}, -\sin\phi\,\dot{\phi}, 0), \quad (0, 0, 0),$$

respectively, where $\dot{\phi}$ stands for $\frac{d\phi}{dt}$. The equations of definition (26.4) give, then,

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = \dot{\phi}. \tag{26.7}$$

More generally, let the orientation of the moving frame $OXYZ$ with respect to the fixed frame $P\xi\eta\zeta$ at any instant t be given by the Eulerian angles (θ, ϕ, ψ). We have, then, from (23.9), the table of direction cosines

	$P\xi$	$P\eta$	$P\zeta$
OX	$\cos\theta\cos\phi\cos\psi$ $- \sin\phi\sin\psi$	$\cos\theta\sin\phi\cos\psi$ $+ \cos\phi\sin\psi$	$-\sin\theta\cos\psi$
OY	$-\cos\theta\cos\phi\sin\psi$ $- \sin\phi\cos\psi$	$-\cos\theta\sin\phi\sin\psi$ $+ \cos\phi\cos\psi$	$\sin\theta\sin\psi$
OZ	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$

(26.8)

From these we readily find

$$\begin{aligned} \dot{\mathbf{i}} &= (\dot\psi + \cos\theta\,\dot\phi)\mathbf{j} - (\dot\theta\cos\psi + \sin\theta\sin\psi\,\dot\phi)\mathbf{k}, \\ \dot{\mathbf{j}} &= -(\dot\psi + \cos\theta\,\dot\phi)\mathbf{i} + (\dot\theta\sin\psi - \sin\theta\cos\psi\,\dot\phi)\mathbf{k}, \\ \dot{\mathbf{k}} &= (\cos\psi\,\dot\theta + \sin\theta\sin\psi\,\dot\phi)\mathbf{i} - (\dot\theta\sin\psi - \sin\theta\cos\psi\,\dot\phi)\mathbf{j}, \end{aligned} \tag{26.9}$$

so that, by (26.4),

$$\begin{aligned} \omega_x &= (\mathbf{k}.\dot{\mathbf{j}}) = \dot\theta\sin\psi - \sin\theta\cos\psi\,\dot\phi, \\ \omega_y &= (\mathbf{i}.\dot{\mathbf{k}}) = \dot\theta\cos\psi + \sin\theta\sin\psi\,\dot\phi, \\ \omega_z &= (\mathbf{j}.\dot{\mathbf{i}}) = \dot\psi + \cos\theta\,\dot\phi. \end{aligned} \tag{26.10}$$

These formulas show that the vector $\boldsymbol{\omega}$ is the sum of a vector $\dot\phi$ along the fixed $\overrightarrow{P\zeta}$ axis, a vector $\dot\theta$ along the line of nodes $\overrightarrow{ON}$, and a vector $\dot\psi$ along the axis $\overrightarrow{OZ}$ of the moving frame. From this we have the presentation of the vector $\boldsymbol{\omega}$ in the fixed frame $P\xi\eta\zeta$ as

$$\begin{aligned} \omega_\xi &= -\dot\theta\sin\phi + \sin\theta\cos\phi\,\dot\psi, \\ \omega_\eta &= \dot\theta\cos\phi + \sin\theta\sin\phi\,\dot\psi, \\ \omega_\zeta &= \dot\phi + \cos\theta\,\dot\psi. \end{aligned} \tag{26.11}$$

The formulas (26.11) can be derived immediately from the formulas (26.10) by writing $-\theta$ for θ, $-\psi$ for ϕ, and $-\phi$ for ψ, and then changing the sign of each expression (see § 23).

27. The velocity vector. If M is any point, we may denote the vectors $\overrightarrow{OM}$ and $\overrightarrow{PM}$ by $\mathbf{r}$ and $\boldsymbol{\rho}$ respectively. Then $\dot{\boldsymbol{\rho}}$ is known as the velocity of the point M with respect to the frame $P\xi\eta\zeta$ or, if we wish, as the absolute velocity of the point M. Similarly, $\frac{d\mathbf{r}}{dt}$ is known as the velocity of the point M with respect to the frame $OXYZ$ or as the relative velocity of M. Denoting the vector $\overrightarrow{PO}$ by $\boldsymbol{\alpha}$, we have

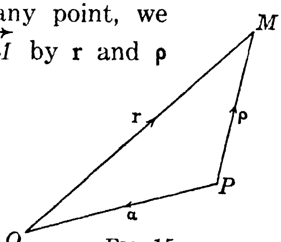

FIG. 15

$$\boldsymbol{\rho} = \boldsymbol{\alpha} + \mathbf{r};$$

whence, by (26.6),

$$\dot{\boldsymbol{\rho}} = \dot{\boldsymbol{\alpha}} + \dot{\mathbf{r}} = \dot{\boldsymbol{\alpha}} + \frac{d\mathbf{r}}{dt} + [\boldsymbol{\omega}.\mathbf{r}]. \tag{27.1}$$

If, now, the moving point happens to be rigidly attached to the $OXYZ$ frame, its relative velocity $\frac{d\mathbf{r}}{dt}$ is zero, and (27.1) reduces to $\dot{\boldsymbol{\alpha}} + [\boldsymbol{\omega}.\mathbf{r}]$. Writing, then,

$$\dot{\boldsymbol{\alpha}} + [\boldsymbol{\omega}.\mathbf{r}] = \dot{\boldsymbol{\rho}}_d, \tag{27.2}$$

we may call $\dot{\boldsymbol{\rho}}_d$ the *drag velocity* of the point M, since it is the absolute velocity of that point of the reference frame $OXYZ$ which is momentarily in coincidence with the moving point M. The equations (27.1) and (27.2) give

$$\mathbf{v}_a = \mathbf{v}_d + \mathbf{v}_r, \tag{27.3}$$

where $\mathbf{v}_a$, $\mathbf{v}_d$, and $\mathbf{v}_r$ denote, respectively, the absolute, drag, and relative velocities of the point M. The point M is usually said to possess simultaneously the two velocities $\mathbf{v}_d$ and $\mathbf{v}_r$, and the result (27.3) is known as the law of composition of velocities. It may evidently be extended to take care of the case where M possesses simultaneously several velocities. Thus let us suppose that we have a "fixed," or "absolute," reference frame F_{n+1}, and a series of "moving," or "relative," frames $F_1, F_2, \cdots, F_n$. Let us denote by $\mathbf{v}_r$ the velocity of a point M with respect to F_1, and by $\mathbf{v}_1$ the velocity, relative to the frame F_2, of that point M_1 of F_1 which happens to be momentarily in coincidence with M; in general, let us denote by $\mathbf{v}_s$ the velocity relative to F_{s+1} of that point M_s of F_s which happens to be momentarily in coincidence with M. Then the point M is said to possess simultaneously the $n + 1$ velocities $\mathbf{v}_r, \mathbf{v}_1, \cdots, \mathbf{v}_n$, and its absolute velocity is given by the equation

$$\mathbf{v}_a = \mathbf{v}_r + \mathbf{v}_1 + \cdots + \mathbf{v}_n. \tag{27.4}$$

If M is permanently in coincidence with M_1, we have $\mathbf{v}_r = \mathbf{0}$, and the drag velocity $\mathbf{v}_d$ is given by the equation

$$\mathbf{v}_d = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n. \tag{27.5}$$

The equation (27.2) shows that the drag velocity of any point M of a moving frame is the vector moment about that point of the localized-vector system $(\boldsymbol{\omega}, \dot{\boldsymbol{\alpha}})$ (see (18.3)). If $\boldsymbol{\omega} = \mathbf{0}$, the localized-vector system is a couple; and in this case every point of the moving frame has the same velocity $\dot{\boldsymbol{\alpha}}$, so that the moving frame is said to have, at the instant in question, a velocity of *translation without rotation* with respect to the fixed frame. The direction and magnitude of the common drag velocity of the various points of the moving frame are given by the vector $\dot{\mathbf{a}}$ of the couple.

If the invariant $(\boldsymbol{\omega}.\dot{\mathbf{a}})$ of the localized-vector system is zero, the system is equivalent to a single segment (see § 19). In particular, if $\dot{\mathbf{a}} = 0$ (that is, if the origin O of the moving frame has zero velocity relative to the fixed frame), the single equivalent segment passes through the origin O, and its axis lies along that representative segment of $\boldsymbol{\omega}$ whose initial point is at O. The drag velocity of any point M of the moving frame is given by

$$\mathbf{v}_d = [\boldsymbol{\omega}.\mathbf{r}], \tag{27.6}$$

and the frame is said to possess an *angular velocity* $\boldsymbol{\omega}$ with respect to the fixed frame $P\xi\eta\zeta$. From our results on the composition of *coplanar* localized vectors (see § 19), we see that if a frame F_1 has an angular velocity $\boldsymbol{\omega}_1$ with respect to a frame F_2, which has itself an angular velocity $\boldsymbol{\omega}_2$ with respect to a "fixed" frame F, then the frame F_1 has an "absolute" angular velocity $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with respect to the fixed frame F. The frame F_1 is usually said to possess simultaneously the two angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, and our result may be stated in the form that angular velocities are compounded or added according to the vector law. It is, of course, expressly understood that the origins of the three frames have no relative velocity. The frames may therefore be supposed, without any lack of generality, to have a common origin.

Since any localized-vector system is equivalent to a single segment and a couple whose plane is perpendicular to the segment, we see that, at any instant, the drag velocities of the various points of a moving frame may be obtained by combining an angular velocity about a given line with a velocity of translation parallel to this line. In other words, the motion at any instant of the various points of a given frame with respect to any other frame is a screw motion. The pitch h of the screw is given by

$$h = \frac{(\boldsymbol{\omega}.\dot{\mathbf{a}})}{\omega^2}, \quad \text{[See (21.4)]} \tag{27.7}$$

and the axis of the screw is given by

$$[\mathbf{r}_{oa}.\boldsymbol{\omega}] = \dot{\mathbf{a}} - h\boldsymbol{\omega}, \tag{27.8}$$

where a is any point on the axis of the screw. When $h = 0$ we have a motion of rotation without translation.

28. The angular velocity vector in orthogonal curvilinear coördinates. We shall first consider the case of space polar coördinates. The moving frame $OXYZ$ has at each point O, whose space polar coördinates are (r, θ, ϕ), the directions of the space polar coördinate lines through O. We have, then, the following table of direction cosines (see (6.4)):

	$P\xi$	$P\eta$	$P\zeta$
OX	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
OY	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
OZ	$-\sin\phi$	$\cos\phi$	0

It follows at once that

$$\begin{aligned} \dot{\mathbf{i}} &= \dot{\theta}\,\mathbf{j} + \dot{\phi}\sin\theta\,\mathbf{k}, \\ \dot{\mathbf{j}} &= -\dot{\theta}\,\mathbf{i} + \dot{\phi}\cos\theta\,\mathbf{k}, \\ \dot{\mathbf{k}} &= -\dot{\phi}\sin\theta\,\mathbf{i} - \dot{\phi}\cos\theta\,\mathbf{j}; \end{aligned}$$

whence we derive

$$\begin{aligned} \omega_x &= (\mathbf{k}.\dot{\mathbf{j}}) = \dot{\phi}\cos\theta, \\ \omega_y &= (\mathbf{i}.\dot{\mathbf{k}}) = -\dot{\phi}\sin\theta, \\ \omega_z &= (\mathbf{j}.\dot{\mathbf{i}}) = \dot{\theta}, \end{aligned} \tag{28.1}$$

showing that the angular velocity vector $\boldsymbol{\omega}$ is equal to the sum of a vector $\dot{\phi}$ along the polar axis and a vector $\dot{\theta}$ along a line perpendicular to the meridian plane in the sense of ϕ increasing.

It is now easy to find the presentation of the absolute velocity of the point $O = (r, \theta, \phi)$ in a frame which has a fixed origin, but whose axes are always parallel to those of the frame $OXYZ$. The formula (27.1) reduces to $\mathbf{v}_a = \frac{d\mathbf{r}}{dt} + [\boldsymbol{\omega}.\mathbf{r}]$, where $\mathbf{r}$ has the presentation $(r, 0, 0)$ in our relative frame. It follows that the relative presentation of the absolute velocity of O is

$$\left(\frac{dr}{dt}, r\dot{\theta}, r\sin\theta\,\dot{\phi}\right). \tag{28.2}$$

For a general system of orthogonal curvilinear coördinates (see § 6), we choose the axes of the fixed frame parallel to those of the moving frame at a definite instant when the latter have the directions of the curvilinear coördinate lines through an arbitrary point (α, β, γ). The direction cosines c_r^s which give the relative orientation of the fixed and moving frames have, therefore, at the point (α, β, γ), the values

$$c_r^r = 1, \quad c_r^s = 0. \quad (r \neq s) \qquad \text{[By (6.15)]}$$

Denoting the Cartesian coördinates of any point with respect to the fixed frame by (ξ, η, ζ), we have

$$\dot{c}_r^s = \frac{\partial c_r^s}{\partial \xi}\dot{\xi} + \frac{\partial c_r^s}{\partial \eta}\dot{\eta} + \frac{\partial c_r^s}{\partial \zeta}\dot{\zeta},$$

where the dots denote, as usual, time differentiations. Now at the point $O = (\alpha, \beta, \gamma)$ the quantities $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ present in the fixed frame, and therefore in the moving frame (since the latter has at O the same directions as the fixed frame), the absolute velocity of the moving origin. If the curve traced out by the moving origin has equations of the type

$$\alpha = \alpha(\xi, \eta, \zeta), \quad \beta = \beta(\xi, \eta, \zeta), \quad \gamma = \gamma(\xi, \eta, \zeta),$$

where (ξ, η, ζ) are functions of the time, we have three equations of the type

$$\dot{\alpha} = \frac{\partial \alpha}{\partial \xi}\dot{\xi} + \frac{\partial \alpha}{\partial \eta}\dot{\eta} + \frac{\partial \alpha}{\partial \zeta}\dot{\zeta}.$$

It follows from the results (6.17), as there (x_1, x_2, x_3) play the rôles taken here by (ξ, η, ζ), that, at the point O, $\dot{\alpha} = h_1\dot{\xi}$. We find in this way that the presentation of the absolute-velocity vector in the moving frame is

$$v_\alpha = \frac{\dot{\alpha}}{h_1}, \quad v_\beta = \frac{\dot{\beta}}{h_2}, \quad v_\gamma = \frac{\dot{\gamma}}{h_3}. \tag{28.3}$$

Since (ξ, η, ζ) play here the rôle taken by (x_1, x_2, x_3) in § 6, the values of the partial derivatives $\left(\frac{\partial c_r^s}{\partial \xi}, \frac{\partial c_r^s}{\partial \eta}, \frac{\partial c_r^s}{\partial \zeta}\right)$ at the point O follow from the results given in (6.19), (6.20), (6.23), (6.28), and (6.29). To evaluate $\boldsymbol{\omega}$ we write

$$(\dot{c}_r^s)_0 = \left(\frac{\partial c_r^s}{\partial \xi}\dot{\xi} + \frac{\partial c_r^s}{\partial \eta}\dot{\eta} + \frac{\partial c_r^s}{\partial \zeta}\dot{\zeta}\right)_0; \tag{28.4}$$

and upon writing $(\dot{\xi})_0 = v_\alpha$, etc. we find, for example,

$$(\dot{c}_3^2)_0 = -(\dot{c}_2^3)_0 = \left[h_2h_3\left(v_\gamma \frac{\partial}{\partial \beta}\frac{1}{h_3} - v_\beta \frac{\partial}{\partial \gamma}\frac{1}{h_2}\right)\right]_0.$$

Two similar expressions may be obtained by permuting the letters (α, β, γ) and the suffixes (1, 2, 3) cyclically. The subscript zero may now be dropped, since O is any point (α, β, γ) whatsoever; and we have the results

$$
\begin{aligned}
\omega_x &= (\mathbf{k}.\dot{\mathbf{j}}) = (\dot{c}_3{}^2)_0 = h_2h_3\left[v_\gamma \frac{\partial}{\partial\beta}\frac{1}{h_3} - v_\beta \frac{\partial}{\partial\gamma}\frac{1}{h_2}\right],\\
\omega_y &= (\mathbf{i}.\dot{\mathbf{k}}) = (\dot{c}_1{}^3)_0 = h_3h_1\left[v_\alpha \frac{\partial}{\partial\gamma}\frac{1}{h_1} - v_\gamma \frac{\partial}{\partial\alpha}\frac{1}{h_3}\right],\\
\omega_z &= (\mathbf{j}.\dot{\mathbf{i}}) = (\dot{c}_2{}^1)_0 = h_1h_2\left[v_\beta \frac{\partial}{\partial\alpha}\frac{1}{h_2} - v_\alpha \frac{\partial}{\partial\beta}\frac{1}{h_1}\right].
\end{aligned} \tag{28.5}
$$

Let us consider, for a moment, the case where the origin O of the moving frame moves along the first coördinate line with unit velocity (see § 27). Then $v_\alpha = 1$, $v_\beta = 0$, $v_\gamma = 0$; and differentiation with respect to the time is equivalent to differentiation with respect to the arc length s_1 along the first coördinate line through O. On denoting by $\boldsymbol{\omega}_1$ the angular velocity of the moving frame when the origin moves along the first coördinate line, we have

$$
\begin{aligned}
(\boldsymbol{\omega}_1)_x &= (\mathbf{k}.\dot{\mathbf{j}})_1 = 0,\\
(\boldsymbol{\omega}_1)_y &= -(\dot{\mathbf{i}}.\mathbf{k})_1 = h_3h_1 \frac{\partial}{\partial\gamma}\frac{1}{h_1},\\
(\boldsymbol{\omega}_1)_z &= (\mathbf{j}.\dot{\mathbf{i}})_1 = -h_1h_2 \frac{\partial}{\partial\beta}\frac{1}{h_1}.
\end{aligned} \tag{28.6}
$$

Similarly, we derive the presentations in the moving frame of the vectors $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$. Now the vector $(\dot{\mathbf{i}})_1$ is of magnitude $\frac{1}{\rho_1}$ and is directed along the principal normal at O to the first coördinate curve (ρ_1 denoting the radius of curvature at O of this curve) (see (5.5)). Let the angle from the y-axis to this principal normal (which, being perpendicular to i, lies in the yz plane) be denoted by θ_1, θ_1 being measured positively around the x-axis. Similarly, denote by θ_2 and θ_3 the angles from the z and x axes to the principal normals to the second and third coördinate curves respectively, θ_2 and θ_3 being measured positively around the y and z axes respectively. We have, then, from (28.6), the results

$$
-h_1h_2\frac{\partial}{\partial\beta}\frac{1}{h_1} = \frac{\cos\theta_1}{\rho_1}, \quad h_3h_1\frac{\partial}{\partial\gamma}\frac{1}{h_1} = -\frac{\cos\left(\theta_1 - \frac{\pi}{2}\right)}{\rho_1} = -\frac{\sin\theta_1}{\rho_1}.
$$

By permuting the suffixes (1, 2, 3) and the letters (α, β, γ), we obtain the corresponding results for the other two coördinate curves. These enable us to write equations (28.5) in the form

$$
\begin{aligned}
\omega_x &= v_\beta \frac{\cos\theta_2}{\rho_2} - v_\gamma \frac{\sin\theta_3}{\rho_3},\\
\omega_y &= v_\gamma \frac{\cos\theta_3}{\rho_3} - v_\alpha \frac{\sin\theta_1}{\rho_1},\\
\omega_z &= v_\alpha \frac{\cos\theta_1}{\rho_1} - v_\beta \frac{\sin\theta_2}{\rho_2}.
\end{aligned} \tag{28.7}
$$

The coördinate lines in a system of orthogonal curvilinear coördinates are lines of curvature on the coördinate surfaces, and the expression $\dfrac{\cos\theta_1}{\rho_1}$ is, by Meusnier's theorem, the curvature at O of that normal section of the second coördinate surface which has the direction of the first coördinate line. We shall denote this curvature, which is one of the principal curvatures at O of the second coördinate surface, by $\dfrac{1}{\rho_{21}}$; and, in general, we shall denote by ρ_{rs} the principal radius of curvature of the rth coördinate surface which corresponds to the direction of the sth coördinate line $(r \neq s)$. Then the equations (28.7) take the form

$$\begin{aligned}\omega_x &= \frac{v_\beta}{\rho_{32}} - \frac{v_\gamma}{\rho_{23}},\\ \omega_y &= \frac{v_\gamma}{\rho_{13}} - \frac{v_\alpha}{\rho_{31}},\\ \omega_z &= \frac{v_\alpha}{\rho_{21}} - \frac{v_\beta}{\rho_{12}}.\end{aligned} \tag{28.8}$$

The convention as to the signs of the principal radii of curvature ρ_{rs} is that ρ_{rs} is positive or negative according as the principal normal to the sth coördinate curve makes an acute or an obtuse angle with the rth coördinate curve. The values of the ρ_{rs} in terms of h_1, h_2, and h_3 follow on comparison of (28.8) and (28.5).

We may apply a similar argument to the case of a moving frame whose origin O is constrained to lie on a given surface, the x and y axes being always directed along the lines of curvature of the surface. If v_γ is zero, we obtain (see (28.7))

$$\omega_x = v_\beta \frac{\cos\theta_2}{\rho_2}, \quad \omega_y = -v_\alpha \frac{\sin\theta_1}{\rho_1}, \quad \omega_z = v_\alpha \frac{\cos\theta_1}{\rho_1} - v_\beta \frac{\sin\theta_2}{\rho_2}. \tag{28.9}$$

Here ρ_1 and ρ_2 are the radii of curvature of the lines of curvature and $\dfrac{\sin\theta_1}{\rho_1}$, $\dfrac{\cos\theta_2}{\rho_2}$, are the two principal curvatures of the surface at O. Denoting the principal radii of curvature by R_1 and R_2 respectively, we have

$$\omega_x = \frac{v_\beta}{R_2}, \quad \omega_y = -\frac{v_\alpha}{R_1}, \quad \omega_z = \frac{v_\alpha \cot\theta_1}{R_1} - \frac{v_\beta \tan\theta_2}{R_2}. \tag{28.10}$$

It is more convenient here to use the angles ϕ_1 and ϕ_2 from the normal to the surface (that is, from the z-axis) to the principal normals to the lines of curvature (ϕ_1 and ϕ_2 being measured positively around the x and y axes respectively). Then $\phi_1 = \theta_1 - \frac{\pi}{2}$, $\phi_2 = \theta_2$, and we have

$$\omega_x = \frac{v_\beta}{R_2}, \quad \omega_y = -\frac{v_\alpha}{R_1}, \quad \omega_z = -\left[\frac{v_\alpha \tan\phi_1}{R_1} + \frac{v_\beta \tan\phi_2}{R_2}\right]. \tag{28.11}$$

The convention as to the signs of R_1 and R_2 is that each of these is positive or negative according as the principal normal to the corresponding line of curvature makes an acute or an obtuse angle with that direction along the normal to the surface which is chosen as the direction of the z-axis of the moving frame.

If we have a system of orthogonal curvilinear coördinates in a plane, and consider a moving frame whose x and y axes have the directions of the coördinate lines through O, the z-axis being, then, always normal to the plane, we have merely to put $v_\gamma = 0$, $\theta_1 = 0$ or π, $\theta_2 = \frac{\pi}{2}$, and we have, from (28.7),

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = \pm\frac{v_\alpha}{\rho_1} \pm \frac{v_\beta}{\rho_2}.$$

In order to remove the ambiguity in signs, let us agree to measure ρ_1 and ρ_2 positively when the corresponding normals to the plane curves have the directions of the second and first coördinate lines respectively, and negatively when they have the opposite directions. Then $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$, and we have

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = \frac{v_\alpha}{\rho_1} - \frac{v_\beta}{\rho_2}. \tag{28.12}$$

The expressions for ρ_1 and ρ_2 in terms of h_1 and h_2 follow from a comparison with (28.5) after putting $v_\gamma = 0$, $h_3 = 1$, and h_1 and h_2 functions of α and β alone. We find

$$\frac{1}{\rho_1} = -h_1h_2\frac{\partial}{\partial\beta}\frac{1}{h_1}, \quad \frac{1}{\rho_2} = -h_1h_2\frac{\partial}{\partial\alpha}\frac{1}{h_2}. \tag{28.13}$$

29. The angular velocity of a frame one of whose axes is always tangent to the curve traced out by the moving origin. If the unit vector along the tangent at O to the curve C traced out by the moving origin O be denoted by $\mathbf{t}_1$, we have already seen (see (5.5)) that

$$\frac{d\mathbf{t}_1}{ds} = \frac{\mathbf{n}}{\rho},$$

where s is the arc distance along C from some convenient origin, ρ is the radius of curvature, and $\mathbf{n}$ is the unit vector along the principal normal at O to the curve C. The vector $\frac{d\mathbf{t}_1}{ds}$ may be called the curvature vector at O of the curve C. Denoting the velocity of O with respect to our fixed frame by v, we have $v = \dot{s}$, and so

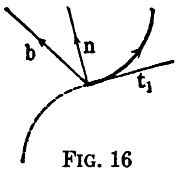

FIG. 16

$$\dot{\mathbf{t}}_1 = \frac{d\mathbf{t}_1}{ds}\dot{s} = \frac{v}{\rho}\mathbf{n}. \tag{29.1}$$

Let us now denote by $\mathbf{b}$ the unit vector along the binormal (that is, the line perpendicular to both the tangent and the principal normal and so related to them that the tangent, principal normal, and binormal form a right-handed frame). Then, from $(\mathbf{b}.\mathbf{t}_1) = 0$, we derive, on differentiation with respect to the time,

$$(\dot{\mathbf{b}}.\mathbf{t}_1) + (\mathbf{b}.\dot{\mathbf{t}}_1) = 0.$$

The term $(\mathbf{b}.\dot{\mathbf{t}}_1) = \frac{v}{\rho}(\mathbf{b}.\mathbf{n})$, by (29.1), equals 0, since $\mathbf{b}$ is perpendicular to $\mathbf{n}$; and so $(\dot{\mathbf{b}}.\mathbf{t}_1) = 0$. Moreover, $(\mathbf{b}.\dot{\mathbf{b}}) = 0$, since $\mathbf{b}$ is a unit vector; and thus we see that $\dot{\mathbf{b}}$ is perpendicular to both $\mathbf{t}_1$ and $\mathbf{b}$. It must therefore have either the same direction as $\mathbf{n}$ or the opposite direction. We shall write

$$\dot{\mathbf{b}} = -\frac{v}{\tau}\mathbf{n}, \tag{29.2}$$

and shall call τ the radius of torsion of the curve C at the point O. According as $\dot{\mathbf{b}}$ has the opposite or the same direction as $\mathbf{n}$, τ is positive or negative. Thus, when τ is positive, the end point of $\mathbf{b}$ is turning positively around $\mathbf{t}_1$ according to the right-hand-screw convention. If we consider the frame whose axes OX, OY, OZ have the directions of $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ respectively, we have

$$\mathbf{i} = \mathbf{t}_1, \quad \mathbf{j} = \mathbf{n}, \quad \mathbf{k} = \mathbf{b};$$

and so

$$\begin{aligned} \omega_x &= -(\dot{\mathbf{k}}.\mathbf{j}) = -(\dot{\mathbf{b}}.\mathbf{n}) = \frac{v}{\tau}, && [\text{By } (29.2)] \\ \omega_y &= -(\dot{\mathbf{i}}.\mathbf{k}) = -(\dot{\mathbf{t}}_1.\mathbf{b}) = -\frac{v}{\rho}(\mathbf{n}.\mathbf{b}) && [\text{By } (29.1)] \\ &= 0, \\ \omega_z &= (\dot{\mathbf{i}}.\mathbf{j}) = (\dot{\mathbf{t}}_1.\mathbf{n}) = \frac{v}{\rho}. && [\text{By } (29.1)] \end{aligned} \tag{29.3}$$

We see, then, that the component, along the tangent, of the angular velocity of the moving frame is $\frac{v}{\tau}$. This explains why $\frac{1}{\tau}$ is called the torsion of the curve. The component in the normal plane has the direction of the binormal and the magnitude $\frac{v}{\rho}$.

If we consider a frame whose x-axis coincides with $\mathbf{t}_1$ but whose y-axis makes an angle θ with the principal normal, the sense of positive measurement of θ being around the tangent *from* the principal normal to the y-axis, this new frame has an angular velocity, relative to the ($\mathbf{t}_1$, $\mathbf{n}$, $\mathbf{b}$) frame, of magnitude $\dot{\theta}$ directed along the tangent (see (26.7)). Hence, by the rule of composition of *concurrent* angular velocities (see § 27), we know that the new frame has, with respect to our fixed frame, an angular velocity whose components along the tangent, principal normal, and binormal, respectively, are

$$\frac{v}{\tau} + \dot{\theta}, \quad 0, \quad \frac{v}{\rho}.$$

The components (ω_x, ω_y, ω_z) along the axes of the moving frame are, accordingly,

$$\begin{aligned} \omega_x &= \frac{v}{\tau} + \dot{\theta}, \\ \omega_y &= \frac{v \sin \theta}{\rho}, \\ \omega_z &= \frac{v \cos \theta}{\rho}, \end{aligned} \tag{29.4}$$

for the angle from the binormal to the z-axis is θ.

30. The acceleration vector. The time derivative, with respect to the fixed frame $P\xi\eta\zeta$, of the absolute-velocity vector is known as the absolute-acceleration vector. Similarly, the time derivative, with respect to the moving frame $OXYZ$, of the relative-velocity vector is known as the relative-acceleration vector. From equation (27.1),

$$\dot{\mathbf{p}} = \dot{\mathbf{a}} + \frac{d\mathbf{r}}{dt} + [\boldsymbol{\omega} . \mathbf{r}],$$

we obtain, on differentiation with respect to the fixed frame,

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + \left(\frac{\dot{d\mathbf{r}}}{dt}\right) + [\dot{\boldsymbol{\omega}} . \mathbf{r}] + [\boldsymbol{\omega} . \dot{\mathbf{r}}];$$

but, by (26.6),
$$\left(\frac{\dot{d\mathbf{r}}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2} + \left[\boldsymbol{\omega} \cdot \frac{d\mathbf{r}}{dt}\right],$$

and, similarly,
$$\begin{aligned} \dot{\boldsymbol{\omega}} &= \frac{d\boldsymbol{\omega}}{dt} + [\boldsymbol{\omega} . \boldsymbol{\omega}] = \frac{d\boldsymbol{\omega}}{dt}, \\ \dot{\mathbf{r}} &= \frac{d\mathbf{r}}{dt} + [\boldsymbol{\omega} . \mathbf{r}]. \end{aligned} \tag{30.1}$$

Hence the absolute-acceleration vector $\ddot{\mathbf{p}}$ is given by

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + \frac{d^2\mathbf{r}}{dt^2} + 2\left[\boldsymbol{\omega} \cdot \frac{d\mathbf{r}}{dt}\right] + \left[\frac{d\boldsymbol{\omega}}{dt} \cdot \mathbf{r}\right] + [\boldsymbol{\omega} . [\boldsymbol{\omega} . \mathbf{r}]]. \tag{30.2}$$

The term $\frac{d^2\mathbf{r}}{dt^2}$ is the relative-acceleration vector; and the terms obtained when $\frac{d\mathbf{r}}{dt} = 0$, namely,

$$\ddot{\mathbf{a}} + \left[\frac{d\boldsymbol{\omega}}{dt}\cdot\mathbf{r}\right] + [\boldsymbol{\omega}.[\boldsymbol{\omega}.\mathbf{r}]],$$

constitute the drag-acceleration vector, that is, the absolute acceleration of that point M_1 of the moving frame which happens to be momentarily in coincidence with the moving point M. There is an additional term, $2\left[\boldsymbol{\omega}\cdot\frac{d\mathbf{r}}{dt}\right]$, in the expression for the absolute-acceleration vector, so that this absolute acceleration is not, like the absolute-velocity vector, the sum of the relative and drag vectors. The additional term is known as the acceleration of Coriolis, and may be denoted by $\mathbf{A}_c$. Denoting the absolute-acceleration, relative-acceleration, and drag-acceleration vectors by $\mathbf{A}_a$, $\mathbf{A}_r$, and $\mathbf{A}_d$ respectively, we have

$$\mathbf{A}_a = \mathbf{A}_r + \mathbf{A}_d + \mathbf{A}_c, \tag{30.3}$$

where

$$\mathbf{A}_d = \ddot{\mathbf{a}} + \left[\frac{d\boldsymbol{\omega}}{dt}\cdot\mathbf{r}\right] + [\boldsymbol{\omega}.[\boldsymbol{\omega}.\mathbf{r}]] \tag{30.4}$$

and

$$\mathbf{A}_c = 2\left[\boldsymbol{\omega}\cdot\frac{d\mathbf{r}}{dt}\right]. \tag{30.5}$$

The vector triple product in the expression (30.4) for the drag acceleration may be expanded, by (3.13), to $(\boldsymbol{\omega}.\mathbf{r})\boldsymbol{\omega} - (\boldsymbol{\omega}.\boldsymbol{\omega})\mathbf{r}$. Hence, if we draw through O the representative segment of the vector $\boldsymbol{\omega}$ and drop on it the perpendicular MQ from M, we may, since

$$\begin{cases}(\boldsymbol{\omega}.\mathbf{r})\boldsymbol{\omega} = \omega^2.\overrightarrow{OQ},\\ (\boldsymbol{\omega}.\boldsymbol{\omega})\mathbf{r} = \omega^2.\overrightarrow{OM},\\ \overrightarrow{OQ} - \overrightarrow{OM} = \overrightarrow{MQ},\end{cases}$$

write

$$[\boldsymbol{\omega}.[\boldsymbol{\omega}.\mathbf{r}]] = (\boldsymbol{\omega}.\boldsymbol{\omega})\,\overrightarrow{MQ} = \omega^2\overrightarrow{MQ}.$$

This gives us

$$\mathbf{A}_d = \ddot{\mathbf{a}} + \left[\frac{d\boldsymbol{\omega}}{dt}\cdot\mathbf{r}\right] + \omega^2\overrightarrow{MQ}. \tag{30.6}$$

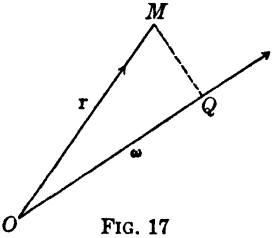

FIG. 17

In particular, when the origin O is at rest in the fixed frame and when the angular velocity is constant, we have

$$\mathbf{A}_d = \omega^2\overrightarrow{MQ}. \tag{30.6 a}$$

In this case the drag acceleration is directed from the point M toward the axis of rotation and is directly proportional to the distance of the point M from the axis of rotation. The drag acceleration is then said to be *centripetal,* and the name is carried over to the general case. Thus the vector sum $\mathbf{A}_d + \mathbf{A}_c$ may be called the centripetal acceleration, so that we have the equation

$$\begin{aligned}\text{Absolute acceleration} &= \text{relative acceleration}\\ &\quad + \text{centripetal acceleration.} \qquad (30.7)\end{aligned}$$

It will be noticed that the acceleration of Coriolis is always perpendicular to the angular-velocity vector and to the relative-velocity vector.

EXAMPLE

Determine the presentation of the absolute-acceleration vector in a frame which has a fixed origin, but whose axes are parallel to the coördinate lines through M of a system of space polar coördinates.

Solution. Here $\mathbf{r}$ has the presentation $(r, 0, 0)$, and $\boldsymbol{\omega}$ the presentation $(\dot{\phi} \cos \theta, -\dot{\phi} \sin \theta, \dot{\theta})$ by (28.1); and we find, as the presentation of the absolute-acceleration vector,

$$\begin{aligned}\Big(\ddot{r} - r\dot{\theta}^2 - r \sin^2 \theta\, \dot{\phi}^2,\ \ \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r \sin \theta \cos \theta\, \dot{\phi}^2,&\\ r \sin \theta\, \ddot{\phi} + 2\, r \cos \theta\, \dot{\theta}\, \dot{\phi} + 2 \sin \theta\, \dot{r}\dot{\phi}\Big).&\end{aligned} \qquad (30.8)$$

On making ϕ constant, so that the motion lies entirely in a meridian plane, we obtain the acceleration components along and perpendicular to the radius vector in planar motion as

$$\left[\ddot{r} - r\dot{\theta}^2,\ \ \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]. \qquad (30.9)$$

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EXERCISES

1. The position of a point is given by the perpendiculars (ξ, η) on two fixed lines containing an angle α. Prove that the component velocities in the directions ξ and η are $\frac{\dot{\xi} + \dot{\eta}\cos\alpha}{\sin^2\alpha}$ and $\frac{\dot{\eta} + \dot{\xi}\cos\alpha}{\sin^2\alpha}$ respectively.

2. A particle describes the circle $r = 2\,a\cos\theta$, the component of acceleration toward the origin being always zero. Show that the transverse component varies as $\operatorname{cosec}^5\theta$.

3. A particle describes an ellipse so that its acceleration is always directed toward the center of the ellipse. Show that the magnitude of the acceleration is proportional to the distance of the particle from the center of the ellipse.

4. A particle describes an equiangular spiral $r = e^{a\theta}$ so that its acceleration is always directed toward the origin. Show that the magnitude of the acceleration is proportional to $\frac{1}{r^3}$.

5. Prove that any conic may be described by a particle with an acceleration always at right angles to the transverse axis and varying inversely as the cube of the distance from it.

6. The radii vectors from two fixed points, separated by a distance c, to the position of a particle are r_1 and r_2, and the velocities in these directions are u_1 and u_2. Prove that the accelerations in the same directions are $\dot{u}_1 + \frac{\frac{1}{2}u_1u_2(r_1^2 - r_2^2 + c^2)}{r_1^2 r_2}$ and $\dot{u}_2 + \frac{\frac{1}{2}u_1u_2(r_2^2 - r_1^2 + c^2)}{r_1 r_2^2}$. The motion is supposed to take place in one plane.

CHAPTER III

DYNAMICS

31. Newton's first fundamental law of dynamics. In order to discuss mathematically the motions of one or more material bodies, it is convenient to introduce the concept of a "material particle," by which is meant a geometric point endowed with the general properties of matter in so far as motion is concerned. In order to discuss the properties of solids, liquids, or gases the postulate is made that they may be considered as composed of material particles. By a series of experiments performed with small spherical bodies, observing such phenomena as their rolling motion on a horizontal table, with friction minimized as much as possible, or their motions when used as pendulums, Galileo and Newton were led to make the following postulate in regard to the motion of a small material particle:

A particle left to itself will maintain its velocity unchanged in direction and magnitude.

This postulate is known variously as Newton's first law of dynamics and as Kepler's principle. It is a pure idealization from observed facts, since a particle can never be "left to itself," that is, removed from the influence of surrounding matter. Furthermore, it presupposes a framework of reference and a method of assigning numbers to time intervals, since it uses the term "velocity." The actual experiments were made with reference to a frame attached to the earth, and with a time scale based on the assumption that the apparent daily motion of the stars is a uniform one. The idealization, however, postulates merely the existence of a framework of reference and of a time scale for which the law holds. If we are led, for example, by use of the postulate, stated for a framework attached to the earth, to results which conflict with experience, we should not at once abandon the postulate but should try to find a new reference frame, and similarly (if necessary) a new time scale, for which the postulate leads to results in harmony with experience.

The property of a particle by which it maintains, when left to itself, a constant vector velocity with respect to a given frame is called *inertia*; and the framework of reference is called an *inertial frame*. The time scale by means of which velocity is defined is called a *Newtonian time scale*. It is found that when a frame attached to the fixed stars, and a time scale based on the assumed uniformity of the length of a sidereal day, are used as an inertial frame and a Newtonian time scale respectively, the results of dynamical theory are, save for a few isolated instances, in agreement with experience.

32. Newton's second fundamental law. When a particle possesses an acceleration relative to an inertial frame, this acceleration is attributed to the influence of other particles, and a force is said to be acting on the particle. Illustrations from experience are the acceleration of a body falling toward the earth, the effect of a magnet upon a piece of iron, the impact of one billiard ball upon another. The simplest case to consider is that of two particles only. Newton concluded from his observations, and those of others, that if, in this case, there is an acceleration of one particle, there is also an acceleration of the other, the two accelerations being directed along the line joining the two particles, but in opposite directions. (If the acceleration of each particle is directed toward the other, we speak of an attraction between the two particles; whereas if it is away from the other particle, we speak of a repulsion between the particles.) As to the numeric values of the two accelerations, Newton concluded that their ratio was a constant for any two given particles, regardless of the condition of motion of the particles or of the cause of their interaction, — impact, gravitational action, etc. He proposed, therefore, to assign a number m_1 to one particle and to define a corresponding number m_2 for the other by the equation

$$m_1\mathbf{A}_1 = -m_2\mathbf{A}_2. \tag{32.1}$$

In this manner a number m may be determined for any particle in terms of the number assigned any standard particle. Consequently Newton postulated that there is a number characteristic of each particle (which number he called a measure of its *mass*) such that in the case of the interaction of two particles $m_1\mathbf{A}_1 = -m_2\mathbf{A}_2$. (It should be noted that this postulate contains the corollary that if m_2 and m_3 are determined by experiments dealing with the interaction of the second and third bodies with the standard body, then, by interaction between these two bodies themselves, the same ratio $\frac{m_2}{m_3}$ would be obtained.)

A further postulate, based upon observations, is that if there is simultaneous interaction between a particle of mass m and two other particles of masses m_1 and m_2, such that, if there were separate interactions, the accelerations would be given by

$$m\mathbf{A}' = -\, m_1\mathbf{A}_1, \quad m\mathbf{A}'' = -\, m_2\mathbf{A}_2,$$

then, in the joint interaction, the acceleration of the particle of mass m is $\mathbf{A} = \mathbf{A}' + \mathbf{A}''$, a vector summation. The product $m\mathbf{A}$ deserves a name, and "kinetic reaction" has been proposed.

The cause of the acceleration of the particle is associated with the presence of the other particles, and we say that there is a "force acting on" the particle. This is simply a description, not in any sense an explanation. Owing to a definite force — that is, to a definite cause — a particle of mass m experiences an acceleration $\mathbf{A}$. The question arises, What acceleration $\mathbf{A}'$ would a particle of mass m' experience if subjected to the same force? The postulate is made that $m'\mathbf{A}' = m\mathbf{A}$. This is in accord with experience. The simplest type of experiment to illustrate it is furnished by Atwood's machine, which consists of a delicately balanced pulley, with its axis horizontal, over whose grooved rim runs a cord suspending weights at its two ends. If these weights are equal, the system is in equilibrium. If, now, a small weight, called a rider, is added to one of the two balanced weights, the system will be given an acceleration. The cause of the acceleration is the weight of the rider. The mass experiencing the acceleration is that of the two balanced weights and that of the rider, neglecting the effect of the rotating pulley. If a different set of balanced weights is used, and the same rider is added, the cause of the acceleration is the same as before, but the mass experiencing the acceleration is different. If the accelerations are measured for the two cases, the results are in accord with the postulate, namely,

$$m\mathbf{A} = m'\mathbf{A}',$$

where the total mass accelerated in the first case is m and in the second case m'.

The proper description of this property of a particle, or of a set of particles, by virtue of which an acceleration is produced in another particle, is to be found by measuring the mass of some one particle and its acceleration when subjected to this cause of acceleration, and multiplying the two. In other words, a vector which has the direction of the acceleration of the particle, and whose magnitude is the product of the magnitude of this acceleration by the mass of the particle, describes the cause of the acceleration. This vector may be written $\mathbf{F}$ and named "force." From what was postulated above, if any particle of mass M is subjected to this force, the acceleration produced will be given by

$$\mathbf{F} = M\mathbf{A}. \tag{32.2}$$

That is,

The force acting on a particle equals in amount and direction its kinetic reaction.

It is all-important to realize that this is an equation stating the equality of two vectors, and not merely a definition of one vector in terms of another. One of the vectors, $\mathbf{F}$, depends on the interaction between the particle and the other particles of the system; the other, $M\mathbf{A}$, depends on the properties of the particle itself. The statement in italics above is equivalent to Newton's first two laws of dynamics.

In view of the postulate made above as to the independence of the causes of acceleration, it follows that if a particle of mass M is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, the acceleration $\mathbf{A}$ assumed by the particle is given by the equation

$$M\mathbf{A} = \mathbf{F}_1 + \mathbf{F}_2; \tag{32.3}$$

in other words, the forces are compounded by vector addition. A special case of this arises when, as the result of the action of two forces, the particle remains at rest or else continues to move with uniform velocity. Then $\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$, or $\mathbf{F}_1 = -\mathbf{F}_2$. Thus the magnitude of a force may be determined by balancing it against a force whose magnitude is known. If a particle is allowed to hang freely suspended by a string, it is under the action of two forces: one is vertically down (for if the string is cut, the particle will fall vertically, with an acceleration);

the other is vertically up, owing to the tension of the string. Experience shows that the acceleration of a particle falling freely is a constant at any point of the earth's surface, independent of the nature or mass of the particle. Let us call this acceleration g. Then the magnitude of the downward force acting on the particle is mg, if m is the mass of the particle. This furnishes an easy method of comparing masses of different particles and therefore of determining the mass of any particle in terms of that of a standard particle. Mass, when determined in this way, is often called *gravitational mass.* The most delicate experiments support the theory that mass factors determined by different types of forces (magnetic, electric, elastic, gravitational, etc.) are always the same, — a theory usually referred to as the theory of the identity of gravitational and *inertial* mass. It is a remarkable fact that gravitational mass seems to be independent of the physical state; for example, the gravitational mass of a certain quantity of liquid is unaltered by solidification or vaporization of the liquid.

33. The third fundamental law of dynamics. It may be of interest to describe more fully Newton's deduction of the concept of mass, because the experiments upon the impact of spheres led also to another principle in dynamics. Newton and others performed numerous experiments upon the impact of two spherical bodies of small size moving in the same horizontal line, measuring their velocities before and after impact. Let u_1 and u_2 denote the components along the line of centers of the velocities of the two bodies before impact, and u'_1 and u'_2 their components in the same direction after impact. Then it is found that the ratio $\frac{(u'_2 - u_2)}{(u_1 - u'_1)}$ is approximately constant for the same pair of bodies and is, in fact, equal to the ratio $\frac{m_1}{m_2}$ of the gravitational masses of the two bodies. The result of the experiments may be expressed by the equation

$$m_1u_1 + m_2u_2 = m_1u'_1 + m_2u'_2. \qquad (33.1)$$

The velocity of a particle of mass m being represented by $\mathbf{v}$, the vector $m\mathbf{v}$ was called by Newton the "quantity of motion" of the particle, and is now known as the momentum of the particle. We may therefore generalize the result (33.1) by the statement that *the total momentum of the two bodies is unaltered by the impact.* If we write this result in the form

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \text{constant},$$

and differentiate it with respect to the time, we get

$$m_1\mathbf{A}_1 + m_2\mathbf{A}_2 = 0, \tag{33.2}$$

or

$$m_1\mathbf{A}_1 = - m_2\mathbf{A}_2, \tag{33.3}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are the accelerations of the two particles whose masses are m_1 and m_2 respectively. Newton explained this result by assuming that during the impact of the two bodies each body exerted, at each instant, a force upon the other, and by postulating that these forces are always equal and opposite. He called the force exerted by one body on the other its "action," and the force exerted by the second body on the first its "reaction," stating his third fundamental law in the form

Action and reaction are equal and opposite.

The action $\mathbf{F}$ of the first body on the second is in general a variable function of the time throughout the duration of the impact, and we shall indicate this by writing it as $\mathbf{F}(t)$. Similarly, on denoting the reaction by $\mathbf{R}(t)$, Newton's third law states that

$$\mathbf{F}(t) + \mathbf{R}(t) = 0.$$

Here $\mathbf{F}(t) = m_2\mathbf{A}_2$, by the second law; and we obtain, on integration with respect to the time throughout the duration of the impact,

$$\int_T^{T'} \mathbf{F}(t)dt = m_2(\mathbf{v}'_2 - \mathbf{v}_2). \tag{33.4}$$

The force $\mathbf{F}(t)$ is said to be an impulsive force and its time integral is called its *impulse.* Equation (33.4) tells us that when an impulsive force acts upon a particle, its impulse is equal to the change of momentum of the particle. Newton's experiments showed that the impulse of the action is equal and opposite to the impulse of the reaction, and this led him to postulate that action and reaction themselves are equal and opposite. Equation (33.4) may be used to define the *mean impulsive force,* where this is the quotient of the change of momentum by the time during which the impulsive force acts. For example, the time required to bring a bullet to rest on entering a board is found by dividing the momentum of the bullet by the mean force of opposition of the board.

Newton generalized the application of the third law so that it was regarded as valid for any action between two bodies. Thus, if one particle exerts a force of attraction **F** upon a second, this second will exert a force **R** upon the first, and the vector sum of **F** and **R** is zero. In other words, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are the accelerations which each particle possesses by virtue of the presence of the other, the vectors $m_1\mathbf{A}_1$ and $m_2\mathbf{A}_2$ are equal and opposite. Thus, if we draw the representative segments of these vectors whose initial points are at the positions of the masses m_1 and m_2 respectively, the segments are equal, parallel, and oppositely directed. Experiments on the rotation of a rigid body led to the adoption of the additional postulate that they lie along the same straight line, the join of the two particles. If a rigid body is pivoted on a fixed axis, a change of angular velocity may be produced by a force acting on the body if the line of action of the force does not intersect the axis of rotation. A force parallel to the axis does not affect the rotation. It is found by experiment that if we regard forces as localized vectors whose axes are their lines of action, two forces have the same effect upon the rotation of a rigid body if their moments about the axis upon which the body is pivoted are the same. In order to account for the fact that the gravitational forces between the various particles of a rigid body do not make the body rotate about any axis, it is postulated that the gravitational action and reaction between two particles are equal and opposite, *the action and reaction vectors being regarded as localized vectors.* They must therefore have the join of the two particles as their common line of action.

As it is essential that the student should have a clear idea as to the exact content of Newton's three fundamental laws, we propose to give here a brief restatement of the foregoing remarks, indicating at the same time a postulational method of laying the foundations of the theory.

1. We must have the idea of a particle — necessarily an undefined term.

2. We consider the existence of this particle either "by itself" or in the presence of other particles or collections of particles.

3. If the particle has an acceleration relative to what we have called an inertial frame, this acceleration is attributed to the presence of other particles or collections of particles.

4. We postulate the independence of the effects of the presence of different influencing particles; that is, $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$.

5. Taking the simplest case, that of the impact of two particles moving in the same straight line, we write

$$\frac{m_2}{m_1} = \frac{v'_1 - v_1}{v_2 - v'_2},$$

since experiment shows that this ratio is the same for two given particles, that is, is independent of their velocities. We postulate that a positive number m may be assigned without ambiguity to any one particle, for example, by assigning a positive number (say, unity) to any one standard body. We postulate also that these numbers are additive, and we call the number assigned in this way to each body the mass of that body. It follows from this definition that $m_1v_1 + m_2v_2$ is unchanged by the impact.

6. We postulate that when the impact of the two particles is oblique, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ remains unchanged; whence $m_1\mathbf{A}_1 = - m_2\mathbf{A}_2$.

7. We next postulate that the two vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ are localized vectors through the points occupied by m_1 and m_2 respectively. (The reasons for making this additional postulate have already been given.)

8. There are many cases in physics where one body receives an acceleration due to the presence of another without any impact taking place; for example, the case of gravitation, the case where one or both of the bodies are electrically charged, etc. We postulate that, in general, $m_1\mathbf{A}_1 = - m_2\mathbf{A}_2$, the vectors being understood to be localized.

9. The product $m_1\mathbf{A}_1$ indicates the presence of the second particle or body. Is it the *measure* of the effect of this presence? That is, if different particles are subjected to the same cause of acceleration, will $m_1\mathbf{A}_1 = m_2\mathbf{A}_2$? The postulate is made that this is the case. It is not easy to give a simple illustration, because, in general, the cause of the acceleration is a *mutual* action between the two particles; that is, it depends numerically upon properties of both. But consider, as in Atwood's machine, a definite rider added to one of a pair of balanced weights, and then again added to one of a different pair: in both these experiments the cause of the acceleration is the weight of the rider. The postulate means that $(2\,m_1 + M)\mathbf{A}_1 = (2\,m_2 + M)\mathbf{A}_2$.

The product $m\mathbf{A}$ is, then, a measure of the cause of the acceleration and is said to measure the "force acting on" the particle. This is written

$$\mathbf{F} = m\mathbf{A}.$$

It is important to realize that this is an equation and not a definition; it expresses the equality between a cause of acceleration, to which numbers may be assigned by certain laws of physics (for example, the law of gravitation), and a product of the mass of a particle by its acceleration. In other words, $\mathbf{F}$ is a quantity whose value depends upon particles and properties apart from the particle whose mass is m and whose acceleration is $\mathbf{A} = \frac{\mathbf{F}}{m}$, as well as upon properties of the particle itself.

Furthermore, $\mathbf{F}$ is to be thought of as a localized vector through the particle whose mass is m, or, if the cause of the acceleration is a second particle, the localized vector joins the two.

Illustrations of some various types of forces which may act on a particle.

a. Gravitation. $\mathbf{F} = \frac{Gm_1m_2\mathbf{r}}{r^3}$, or $F = \frac{Gm_1m_2}{r^2}$;

whence, since $\mathbf{F} = m_1\mathbf{A}_1$, $A_1 = \frac{Gm_2}{r^2}$.

b. Electric force. $\mathbf{F} = \frac{e_1e_2\mathbf{r}}{\epsilon r^3}$,

whence $\mathbf{A}_1 = \frac{e_1e_2\mathbf{r}}{m_1\epsilon r^3}$.

c. Impact. The force varies rapidly, rising from zero to a maximum, and again decreasing to zero. Its value at any instant during the impact is not generally known. Upon integrating with respect to the time during the impact, we get the impulse of the force, which is measured by the change in momentum of the particle. If, instead of taking the time integral of a force, we took its space integral, we should be led to the important idea of energy, as we shall shortly see. Energy is measured by the product of mass by the *square* of the velocity of the particle, and there existed for many years a historic dispute on the question as to whether the effect of an external agency acting upon a particle should be measured by the product of the mass of the particle by its velocity or by the product of its mass by the square of its velocity. The question is whether we shall consider, in addition to the concept of force, the *time* during which it acts or the *distance* through which the particle, on which the force is acting, moves. The former point of view is the more useful for impulsive forces whose time of application is so short that the displacement of the particle on which the force is acting is negligible. For forces acting throughout a considerable interval of time the other viewpoint is more useful.

34. Inertial reference frames and Newtonian time scales. If we have two reference frames, one of which is called the fixed or absolute frame, and the other the moving or relative frame, we have the relation

$$\mathbf{A}_a = \mathbf{A}_r + \mathbf{A}_d + \mathbf{A}_c,$$

connecting the absolute acceleration with the relative, drag, and Coriolis accelerations (see § 30). If the absolute frame is an inertial one, we have, for a particle left to itself, but in an arbitrary position and possessing an arbitrary velocity, $\mathbf{A}_a = \mathbf{0}$. In order that the relative frame should be also an inertial frame, it is necessary and sufficient, from the definition of an inertial frame, that $\mathbf{A}_r = \mathbf{0}$ under the same assumption of arbitrary initial conditions. This gives $\mathbf{A}_d + \mathbf{A}_c = \mathbf{0}$ for all initial positions and velocities. To take a specific case, if we assume the particle to be at the origin of the relative frame, and the relative velocity to be zero, then $\mathbf{A}_d$ reduces to the acceleration of the origin of the moving frame, and $\mathbf{A}_c$ becomes zero. Hence the origin of the moving frame must have zero acceleration with respect to an inertial frame if the moving frame is to be itself an inertial one. Again, the relative velocity $\mathbf{v}_r$ of the moving particle enters $\mathbf{A}_d + \mathbf{A}_c$ only in the term $\mathbf{A}_c$, which is $2[\boldsymbol{\omega} . \mathbf{v}_r]$. As $\mathbf{v}_r$ is varied, $\mathbf{A}_d$ is unaltered; and hence $\mathbf{A}_c$ must be unaltered, since the sum $\mathbf{A}_d + \mathbf{A}_c$ is the zero vector, independently of the value of $\mathbf{v}_r$. This shows that $\boldsymbol{\omega}$ must be zero; and it then follows at once, from the expression for $\mathbf{A}_d$, (30.4), that $\mathbf{A}_d$ vanishes independently of the position of the moving particle. Hence, if the moving frame is to be an inertial one, it must have a motion of uniform velocity without rotation with respect to an inertial frame; and, conversely, any such frame is an inertial one. For such a moving frame $\mathbf{A}_d = \mathbf{0}$ and $\mathbf{A}_c = \mathbf{0}$ for any particle, and we have

$$\mathbf{A}_a = \mathbf{A}_r. \tag{34.1}$$

Hence the measure of any force $\mathbf{F}$ is the same in both inertial frames. No dynamic experiment (that is, one which has to do with the relation of forces to each other) can, therefore, distinguish between the two inertial frames. Neither one of any two inertial frames is more significant than the other; and one may say, with equal propriety, that either of them is "fixed," the other then having a uniform velocity of translation in a straight line. The question as to whether that one of the two frames which we call fixed is "really" fixed is a meaningless one as far as dynamic considerations go. A fixed frame is merely one for which Newton's laws of motion hold, — in other words, an inertial frame. This is known as the Galilean or Newtonian principle of relativity. According to this principle there is no one absolute frame, as far as dynamics is concerned, with respect to which motions may be described. There is, in fact, a triply infinite system of inertial frames, corresponding to the single arbitrary uniform velocity of translation, any one of which is as convenient as any other for the statement of dynamic laws in as simple a form as possible.

Let us now consider the time scale. We shall suppose that we have two time scales; and we shall use the symbols t and τ to denote the measures of any interval of time in the two scales. Granting that the first scale is Newtonian, we ask what are the necessary and sufficient conditions for the second scale also to be Newtonian. If $\mathbf{r}$ is the position vector of a particle in any inertial frame, it is necessary that, when the particle is left to itself $\left(\text{that is, when } \frac{d^2\mathbf{r}}{dt^2} = 0\right)$, $\frac{d^2\mathbf{r}}{d\tau^2}$ should also be zero, and this for arbitrary values of $\mathbf{r}$ and $\frac{d\mathbf{r}}{dt}\cdot$ Now

$$\frac{d\mathbf{r}}{d\tau} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{d\tau}{dt}} \quad \text{and} \quad \frac{d^2\mathbf{r}}{d\tau^2} = \frac{\left(\frac{d\tau}{dt}\frac{d^2\mathbf{r}}{dt^2} - \frac{d\mathbf{r}}{dt}\frac{d^2\tau}{dt^2}\right)}{\left(\frac{d\tau}{dt}\right)^3},$$

so that we must have $\frac{d\mathbf{r}}{dt}\frac{d^2\tau}{dt^2} = \mathbf{0}$ for arbitrary values of $\frac{d\mathbf{r}}{dt}\cdot$ This gives $\frac{d^2\tau}{dt^2} = 0$, or $\tau = at + b$, where a and b are numeric constants. In other words, the time scale τ differs from the time scale t at most by a change in the time origin and in the size of the time unit. This may be expressed by the statement that there is *essentially* only one Newtonian time scale.

35. Non-inertial frames. Dynamic experiments are, of necessity, performed on the earth, and it is natural to use as a reference frame one which is attached to the earth. Such a reference frame is not an inertial one, and it is therefore important to see how forces are measured in a non-inertial frame. If we multiply the equation

$$\mathbf{A}_a = \mathbf{A}_r + \mathbf{A}_d + \mathbf{A}_c$$

by the mass m of the particle under discussion and solve for $m\mathbf{A}_r$, we find

$$m\mathbf{A}_r = m\mathbf{A}_a - m\mathbf{A}_d - m\mathbf{A}_c,$$

or

$$m\mathbf{A}_r = \mathbf{F} - m\mathbf{A}_d - m\mathbf{A}_c, \tag{35.1}$$

where $\mathbf{F}$ is the force acting on the particle. The vector $m\mathbf{A}_r$, which may be termed the relative kinetic reaction of the particle, no longer measures the force $\mathbf{F}$ exerted by the surrounding particles upon the particle under consideration. If, however, we were unaware that the reference frame was a non-inertial one, we should regard $m\mathbf{A}_r$ as measuring the force acting upon the particle; and so we should not be measuring the effect of the absolute force $\mathbf{F}$, which would be observed in an inertial frame, but of $\mathbf{F} + \mathbf{C}$, where

$$\mathbf{C} = - m\mathbf{A}_d - m\mathbf{A}_c. \tag{35.2}$$

It is usual to regard the relative kinetic reaction $m\mathbf{A}_r$ as measuring what is called the *relative force* acting upon the particle. Denoting this by $\mathbf{F}_r$, we have

$$\mathbf{F}_r = \mathbf{F} + \mathbf{C}. \tag{35.3}$$

It is important to notice that $\mathbf{C}$, as defined by (35.2), depends merely on the mass, position, and relative velocity of the particle, together with the motion of the non-inertial frame relative to an inertial one. If these are all maintained the same, $\mathbf{C}$ is unaltered when $\mathbf{F}$, and consequently $\mathbf{F}_r$, are altered. Thus, if we analyze into two sets the particles which are exerting a force upon the particle under consideration, we may write

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2.$$

(This is merely the postulate of the independent action of forces; see § 32.) We have also the two equations

$$\mathbf{F}_{r,1} = \mathbf{F}_1 + \mathbf{C},$$

$$\mathbf{F}_{r,2} = \mathbf{F}_2 + \mathbf{C},$$

the vector $\mathbf{C}$ being the same in each case. These equations show us that $\mathbf{F}_r$ is not, in general, equal to the sum of $\mathbf{F}_{r,1}$ and

$\mathbf{F}_{r,2}$; that is, that *relative forces are not, in general, additive.* This should emphasize the difference between *relative* forces and *inertial* forces. In fact, the use of the term "relative force" is, from many points of view, unfortunate. By the word "force" we consciously or unconsciously understand a mutual property of a set of particles and a single particle, this property being usually brought to our notice by the acceleration of the latter. What has been called, above, the relative force is not a property of the external acting set of particles alone but depends in a very essential way upon the reference frame and upon the motion of the proof particle with respect to the frame.

The equation giving the resultant relative force due to two sets of particles acting simultaneously is

$$\mathbf{F}_r = \mathbf{F} + \mathbf{C} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{C} = \mathbf{F}_1 + \mathbf{F}_{r,2} = \mathbf{F}_2 + \mathbf{F}_{r,1}. \quad (35.4)$$

In other words, the resultant relative force is equal to the sum of the *absolute* force due to one set of particles and the *relative* force due to the other set. This will explain how we can measure absolute forces in a non-inertial frame, such as one attached to the earth. Thus, let us suppose that a particle is at rest in a non-inertial frame under the influence of a certain absolute force $\mathbf{F}_1$. Since the relative force is zero, we have

$$0 = \mathbf{F}_1 + \mathbf{C} = \mathbf{F}_{r,1}.$$

Now let the particle be subjected to the influence of a second absolute force $\mathbf{F}_2$. We have, for the resultant force (relative),

$$\mathbf{F}_r = \mathbf{F}_2 + \mathbf{F}_{r,1} = \mathbf{F}_2. \quad (35.5)$$

In words, the *additional* absolute force is measured by the relative kinetic reaction of the particle. Of course, the original absolute force which was necessary to maintain the particle in a state of rest relative to the non-inertial frame must still be acting on the particle. It is this which serves to neutralize the term $\mathbf{C}$ in the expression for the relative force.

As an example let us suppose, for the moment, that a reference frame attached to the earth may be regarded, to a sufficient degree of approximation, as an inertial frame, and let us consider the case of an observer who is inside a box which is falling freely to the earth under the influence of gravity. Such an observer would not be able to observe the influence of gravity; for a particle "left to itself" inside the box would have no acceleration relative to a reference frame attached to the box. *Omitting gravity,* the forces measured by him would, however, be the same as the forces measured by an observer attached to the earth.

The simplest example of a non-inertial frame, excepting the one which has an acceleration in a straight line relative to an inertial frame, is that which gave the name "centrifugal forces" to the apparent forces due to the fact that the reference frame is not an inertial one. Here the origin O of the moving frame is at rest in an inertial frame, and the angular-velocity vector $\boldsymbol{\omega}$ is constant. We have, then, from (30.6) (see Fig. 17, p. 101),

$$\mathbf{A}_d = \omega^2 \overrightarrow{MQ}, \qquad \mathbf{A}_c = 2\left[\boldsymbol{\omega} \cdot \frac{d\mathbf{r}}{dt}\right]. \tag{35.6}$$

If the particle is at rest relative to the moving frame, $\mathbf{A}_c = 0$, and the centrifugal force is given by

$$\mathbf{C} = -m\mathbf{A}_d = m\omega^2 \overrightarrow{QM}. \tag{35.7}$$

Here Q is the foot of the perpendicular from the point M to the axis of rotation. If there is no absolute force acting on the particle, it experiences, nevertheless, a relative force whose magnitude is $mr\omega^2$, where r is the distance of the particle from the axis of rotation, and whose direction is perpendicular to the axis of rotation and away from it in the plane of the particle and the axis of rotation.

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EXERCISES

1. A particle moves in a smooth circular tube of radius a which rotates about a fixed vertical diameter with uniform angular velocity ω. Prove that if θ is the angular distance of the particle from the lowest point, and if initially it is at rest relative to the tube, with the value α for θ, where $\omega \cos \frac{\alpha}{2} = \left(\frac{g}{a}\right)^{\frac{1}{2}}$, then, at any subsequent time t, $\cot \frac{\theta}{2} = \cot \frac{\alpha}{2} \cosh \left(\omega t \sin \frac{\alpha}{2}\right)$.

2. A particle of unit mass describes an orbit under an attractive force of magnitude P directed toward the origin and a transverse force of magnitude T perpendicular to the radius vector. Prove that the differential equation of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\frac{du}{d\theta}$, where $u = \frac{1}{r}$ and $\frac{dh^2}{d\theta} = \frac{2\,T}{u^3}$.

3. Show that when the orbit of a particle is an ellipse, and the acceleration is always directed toward the center, the magnitude of this acceleration is proportional to the radius vector to the particle, and the magnitude of the velocity is proportional to the length of the diameter conjugate to the diameter through the point.

4. A particle moves in a plane so that its acceleration is always toward a given line and of magnitude proportional to its distance from it; show that its path is a sine curve.

5. If a particle move in a rectangular hyperbola so that its acceleration is always parallel to an asymptote, find the magnitude of this acceleration.

6. A crank OA turns in a plane with uniform angular velocity ω about O and a connecting rod AB pivoted at A communicates rectilinear motion in a fixed direction to a cross-head B. Find the velocity and acceleration of B at any instant; also find its velocity and acceleration relative to the crank-pin A.

7. A particle subjected to a constant absolute force is constrained to move on a plane which turns with uniform angular velocity about a given line. Discuss the motion of the particle, treating in detail the case where the line about which the plane rotates is perpendicular to the constant force.

8. Show that if a particle be subject to a central acceleration of magnitude $r\omega^2$ its path relative to axes rotating in the plane of motion with angular velocity ω will be a circle described with constant angular velocity $2\,\omega$.

CHAPTER IV

THE DYNAMICS OF A PARTICLE; HARMONIC VIBRATIONS

36. Introduction. We shall discuss in this and the following chapters various problems dealing with the motion of a single material particle under the action of various forces. The reference frame may or may not be an inertial one. If it is not an inertial frame, the "centrifugal forces," which owe their origin to the motion of the reference frame with respect to an inertial one, will be supposed to be included in the forces that are given as acting upon the particle. Hence we shall have to solve the second-order differential equation

$$\text{Mass} \times \text{acceleration} = \text{force},$$

or

$$m\mathbf{A}_r = \mathbf{F}_r,$$

where the relative force $\mathbf{F}_r$ is the sum of the absolute force and the centrifugal forces acting upon the particle. We shall omit the subscript r, and shall understand, if no reference is made to the motion of the reference frame, that this frame is an inertial one.

37. The motion of a particle in a straight line. Let us suppose that a particle is acted upon by a force $\mathbf{F}$ whose magnitude is a function only of the distance r from a fixed point O to the position P of the particle, and whose direction is the same as or opposite to that of the vector $\overrightarrow{OP}$. The particle is then said to be under the action of a central force, and the fixed point O is called the center of force. Denoting the vector $\overrightarrow{OP}$ by $\mathbf{r}$, the equation of motion of the particle is

$$m\ddot{\mathbf{r}} = \frac{f(r)\mathbf{r}}{r}, \tag{37.1}$$

where $f(r)$ is the magnitude of the acting force if it has the same direction as $\mathbf{r}$, and the negative of its magnitude if it has the direction opposite to $\mathbf{r}$. It follows, from (37.1), that

$$[\ddot{\mathbf{r}}.\mathbf{r}] = 0.$$

Hence, by (5.4),
$$\frac{d}{dt}[\dot{\mathbf{r}}.\mathbf{r}] = 0;$$

so that
$$[\dot{\mathbf{r}}.\mathbf{r}] = \mathbf{c}, \tag{37.2}$$

where $\mathbf{c}$ is a constant vector. Hence $(\mathbf{c}.\mathbf{r}) = 0$, so that $\overrightarrow{OP}$ lies in the plane through O perpendicular to $\mathbf{c}$. We have, then, the result that *a particle under the action of a central force traces out a plane curve.* The plane in which the path lies passes through O and through the initial position of the particle, and contains the initial direction of motion.

Let us now suppose that the initial velocity has the same direction as, or the opposite direction to, that of $\mathbf{r}$; that is, that $[\dot{\mathbf{r}}.\mathbf{r}]$ initially equals 0 and therefore must always be zero, so that $\dot{\mathbf{r}} = k\mathbf{r}$, where k is a scalar quantity. The position of the particle at any time t being given by the power series

$$\mathbf{r} = (\mathbf{r})_0 + (t - t_0)(\dot{\mathbf{r}})_0 + \tfrac{1}{2}(t - t_0)^2(\ddot{\mathbf{r}})_0 + \cdots,$$

it follows that the particle moves along the straight line through O. Choosing a sense of direction along this straight line, we shall denote by x the displacement OP; so that x is positive when P is on one side of O, and negative when it is on the other. The equation (37.1) now reduces to the scalar equation

$$m\ddot{x} = f(x). \tag{37.3}$$

If $f(x)$ is an odd function, the direction of the acting force will change as the particle passes through O, whereas if $f(x)$ is an even function, the direction of the force will not change on passage through O. If we wish to deal with a force whose magnitude is an even function of x, but whose direction changes on passage of the particle through O, we shall have to discuss the two equations

$$m\ddot{x} = \pm f(x),$$

one of which holds when the particle is on one side of O, and the other when the particle is on the other side of O.

Upon multiplication of (37.3) by $\dot{x}$, and integration with respect to t, we obtain the first integral,

$$\tfrac{1}{2}m\dot{x}^2 = \int^x f(s)ds.$$

This is known as the *energy integral* and is usually written in the form

$$T + V = h, \tag{37.4}$$

where $T = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} mv^2$ is called the kinetic energy of the particle (relative to the reference frame with respect to which the velocity v is measured), $V = -\int^x f(s)ds$ is called the potential energy of the particle (with respect to the same reference frame), and h is known as the energy constant. Since the lower limit in the integral defining V is not specified, V is undefined to the extent of an additive constant. It is to be observed that V has the property that $-\frac{dV}{dx} = f(x)$.

From (37.4) we may at once find t as a function of x. Thus

$$\dot{x}^2 = \frac{2}{m}(h - V),$$

and

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx}{\sqrt{h - V}}, \tag{37.5}$$

where x_0 is the value of x at time $t = t_0$. An inversion of the integral in (37.5) gives us x as a function of t; so that the position of the particle at all times is known, and the problem of motion is solved (for a given lower limit in V and a given h).

A particularly important case in applications of the theory is that in which $f(x) = -n^2x$, n being a real constant, so that the equation of motion is $m\ddot{x} = -n^2x$. Here the acting force is directed toward the center O and is directly proportional to the displacement of the particle from the center of force. Such a force is known as an elastic force. We have $V = \frac{1}{2} n^2x^2$, measured from $x = 0$ (that is, the position for which $f(x) = 0$); and (37.4) takes the form

$$\frac{1}{2} m\dot{x}^2 + \frac{1}{2} n^2x^2 = h,$$

so that h is a positive constant. Writing it in the form $\frac{1}{2} n^2a^2$, we have $\dot{x}^2 = \frac{n^2}{m}(a^2 - x^2)$, giving $\dot{x} = 0$ when $x = \pm a$. It follows that x is constrained to lie in the interval $-a \leqq x \leqq a$; and as $\dot{x}$ vanishes only at the end points, the particle describes the entire interval. Denoting by t_0 a value of t corresponding to $x = 0$, we have, from (37.5),

$$n(t - t_0) = \sqrt{m}\int_0^x \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{m}\ \text{arc}\sin\frac{x}{a};$$

so that

$$x = a \sin \frac{n}{\sqrt{m}}(t - t_0). \tag{37.6}$$

This type of motion is said to be a simple harmonic motion, or vibration, of amplitude a. The motion is periodic, as x, $\dot{x}$, and all the higher derivatives repeat their values after the interval of time $T = \frac{2\pi\sqrt{m}}{n}$. This is known as the *period* of the vibration, and its reciprocal $\nu_0 = \frac{n}{2\pi\sqrt{m}}$, which gives the number of vibrations per unit time, is known as the *frequency* of the simple harmonic motion. A convenient way to remember the formula for the period T is to note that $\frac{n^2}{m}$ is the coefficient of $-x$ in the equation

$$\ddot{x} = -\frac{n^2}{m}x$$

giving the acceleration. Hence

$$T = \frac{2\pi}{\sqrt{\text{coefficient of } - x \text{ in acceleration}}}.$$

It is essential to note that T depends only on the constants of the system, m and n, and not on the initial conditions, amplitude, etc.

We may rewrite (37.6) in the form

$$x = a \sin 2\pi\nu_0(t - t_0) \quad \text{or} \quad a\sin(2\pi\nu_0 t - \delta), \tag{37.6 a}$$

where

$$\delta = 2\pi\nu_0 t_0;$$

and, on introducing $\delta' = \delta + \frac{\pi}{2}$, this may also be written in the form

$$x = a\cos(2\pi\nu_0 t - \delta').$$

The argument $2\pi\nu_0 t - \delta'$ of the cosine is called the phase of the vibration. When the phase is zero, x has its maximum value a; and when $t = 0$, $x = a\cos\delta'$, so that a difference in the value of δ' means that we begin counting time at a different position of the vibrating particle.

It should be noticed that the law of force given by $f(x) = -n^2x$ is of wide applicability. When we have a force which is always directed toward a fixed center O, and which is zero when the particle is at O, the "direct distance" law may be used, provided that the vibrations are of small amplitude, that is, that the values taken by x are small. For we may expand $f(x)$ by means of a power series

$$f(x) = c_1x + c_2x^2 + c_3x^3 + \cdots,$$

there being no constant term since $f(0) = 0$. Moreover, c_1 is negative, since the force is directed toward O; and since, for small values of x, the first term is the dominant term, we may use, as a first approximation, the law of force given by $f(x) = c_1x = -n^2x$, which has been discussed above.

EXERCISE. Discuss the composition of two simple harmonic vibrations of the same frequency in the same straight line. If the amplitudes of the separate vibrations are A_1 and A_2, and the phases $2\pi\nu t - \delta_1$ and $2\pi\nu t - \delta_2$ respectively, the amplitude and phase of the resultant simple harmonic vibration are

$$A = [A_1^2 + A_2^2 + 2A_1A_2 \cos(\delta_2 - \delta_1)]^{\frac{1}{2}}$$

and $\delta = \arctan \dfrac{A_1 \sin \delta_1 + A_2 \sin \delta_2}{A_1 \cos \delta_1 + A_2 \cos \delta_2}$.

Discuss the special cases $\delta_2 - \delta_1 = 0$ or π.

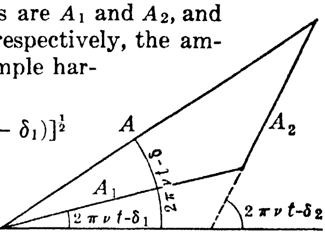


FIG. 18

38. Damped harmonic motion in a straight line. It frequently happens that, in addition to the elastic force $-n^2x$ acting on a particle, there is a force whose magnitude is directly proportional to the velocity of the particle and whose direction is opposite to the direction of motion. Denoting the magnitude of this force by $\pm k\dot{x}$ (the negative sign being used when $\dot{x}$ is negative, and k being positive), the differential equation for x, as a function of t, is

$$m\ddot{x} = -n^2x - k\dot{x},$$

or

$$m\ddot{x} + k\dot{x} + n^2x = 0. \tag{38.1}$$

This is a second-order differential equation with constant coefficients. As equations of this type are of frequent occurrence, it will be useful to give a brief account of the method of solving them. Since (38.1) is a second-order differential equation, it possesses two, *and only two*, distinct solutions, the general solution being a linear combination of these two with constant coefficients. Thus, if x_1 and x_2 denote *any* two solutions of

(38.1) whose ratio is not a constant, the general solution of (38.1) is

$$x = c_1x_1 + c_2x_2,$$

where c_1 and c_2 are constants. It is obvious, however, from the very form of (38.1) that, if x_1 is a solution, so also are $\dot{x}_1, \ddot{x}_1, \cdots$; and, in fact, any order derivative of x_1 with respect to t is a solution. The solutions of (38.1) possess, therefore, the property that any derivative, from the second on, is a linear combination of the solution itself and its first derivative. The simplest function possessing this property of repetition upon differentiation is $x = e^{\alpha t}$, where α is a constant. We might use this as a trial solution, to see if it is possible to satisfy the differential equation by assigning a proper value to α. This would be determined by substituting $e^{\alpha t}$ directly for x in the equation (38.1). It is better, however, to make a change of variable and to write $x = e^{\alpha t}\xi$, where ξ is a new dependent variable. Upon denoting the polynomial

$$m\alpha^2 + k\alpha + n^2$$

by $f(\alpha)$, we find that ξ must satisfy the equation

$$m\ddot{\xi} + (2\,m\alpha + k)\dot{\xi} + (m\alpha^2 + k\alpha + n^2)\xi = 0, \qquad (38.2)$$

or

$$m\ddot{\xi} + f'(\alpha)\dot{\xi} + f(\alpha)\xi = 0. \qquad (38.2\,a)$$

It is obvious that $\xi =$ constant satisfies this equation, provided that $m\alpha^2 + k\alpha + n^2$ or $f(\alpha) = 0$; that is, provided that α is a root of the *characteristic equation*

$$m\alpha^2 + k\alpha + n^2 = 0, \qquad (38.3)$$

and the solutions of (38.1) are $e^{\alpha_1 t}$ and $e^{\alpha_2 t}$, where α_1 and α_2 are two distinct roots of (38.3). If α is a double root of the characteristic equation (38.3), any linear function $\xi = at + b$ (a and b being constants) also satisfies (38.2); for not only is $f(\alpha) = 0$, but $f'(\alpha)$ is also zero. In this case $\alpha = -\dfrac{k}{2\,m}$, and the solutions of (38.1) are $e^{\alpha t}$ and $te^{\alpha t}$.

The argument is perfectly general. If we have a homogeneous linear differential equation of the nth order with constant coefficients

$$a_0x^{(n)} + a_1x^{(n-1)} + \cdots + a_{n-1}\dot{x} + a_nx = 0,$$

we have the characteristic equation

$$a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n = 0.$$

If $\alpha = p$ is a root of this equation of multiplicity r, we have the r distinct solutions $e^{pt}, te^{pt}, \cdots, t^{r-1}e^{pt}$ of the linear homogeneous differential equation. In this way, by considering the different roots of the characteristic equation, we can find n distinct solutions of the differential equation, and the most general solution is a linear combination of these with constant coefficients.

The coefficients of the differential equation, and therefore of the characteristic equation, being assumed real, we know that if a root of the characteristic equation happens to be complex, its conjugate complex quantity is also a root of the characteristic equation. We obtain, then, in the general case, two distinct solutions of the type $t^ke^{(p+iq)t}$ and $t^ke^{(p-iq)t}$. Any linear combination of these can be put in the form $t^ke^{pt}(A \cos qt + B \sin qt)$ or $Ct^ke^{pt} \cos (qt - \delta)$. It is this last form which it is usually most convenient to use.

Returning to our equation $m\ddot{x} + k\dot{x} + n^2x = 0$, we have three distinct cases to consider, according as the roots of the characteristic equation $m\alpha^2 + k\alpha + n^2 = 0$ are

$$\left.\begin{array}{l} \text{(1) conjugate complex quantities,} \\ \text{(2) real and distinct,} \\ \text{(3) real and equal.} \end{array}\right\} \alpha = \frac{-k \pm \sqrt{k^2 - 4\,mn^2}}{2\,m}.$$

(1) Here $k^2 < 4\,mn^2$. Writing $2\,\pi\bar{\nu}_0 = \dfrac{(4\,mn^2 - k^2)^{\frac{1}{2}}}{2\,m}$, the most general solution of the differential equation is

$$x = Ce^{-\frac{kt}{2m}} \cos (2\,\pi\bar{\nu}_0 t - \delta).$$

This is a harmonic motion with steadily decreasing amplitude. Theoretically it would take an infinite time for the amplitude $Ce^{-\frac{kt}{2m}}$ to die down to zero. The accompanying figures show the damping out of the amplitude $Ce^{-\frac{kt}{2m}}$ and the displacement x.

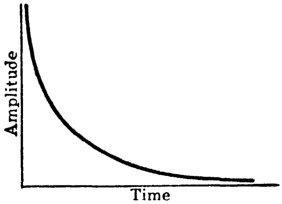


FIG. 19

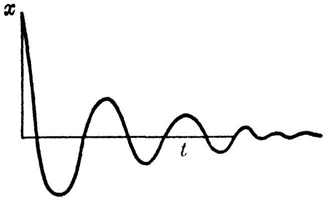


FIG. 20

The positions of zero velocity are given by putting $\dot{x} = 0$, so that

$$\tan (2\,\pi\bar{\nu}_0 t - \delta) + \frac{k}{4\,\pi m\bar{\nu}_0} = 0.$$

They occur, therefore, at intervals of time $\frac{1}{2\bar{\nu}_0}$. The corresponding values of x are positive and negative alternately. Denoting their numeric values by $r_1, r_2, \cdots$, we have

$$\frac{r_1}{r_2} = \frac{r_2}{r_3} = \frac{r_3}{r_4} = \cdots = e^{\frac{k}{4m\bar{\nu}_0}},$$

or

$$\log\frac{r_1}{r_2} = \log\frac{r_2}{r_3} = \cdots = \frac{k}{4m\bar{\nu}_0}. \qquad (38.4)$$

This common logarithmic ratio of successive maxima of the numeric value of x is known as the half logarithmic decrement. The logarithmic decrement itself is the logarithm of the ratio of two amplitudes at phases differing by 2π, or times differing by a complete period. The expression for the logarithmic decrement is therefore $\frac{k}{2m\bar{\nu}_0}$. This may be determined by experiment, and from it follows the value of the damping factor k, since the period $\frac{1}{\bar{\nu}_0}$ may be measured. The frequency $\bar{\nu}_0$ of the damped harmonic vibration is connected with the frequency ν_0 of the undamped vibration ($k = 0$) by the relation

$$\bar{\nu}_0^2 = \frac{1}{4\pi^2}\left(\frac{n^2}{m} - \frac{k^2}{4m^2}\right) = \nu_0^2 - \frac{k^2}{16\pi^2 m^2} = \nu_0^2\left(1 - \frac{k^2}{4mn^2}\right). \qquad (38.5)$$

The effect, then, of the damping, or frictional, force $-k\dot{x}$ is twofold:

(*a*) The amplitude is damped from the constant value C to $Ce^{-\frac{kt}{2m}}$. This effect is of the order of the first power of k. The amplitude is reduced from C to Ce^{-1} in the time $t = \frac{2m}{k}$.

(*b*) The frequency is diminished from the value $\nu_0 = \frac{n}{2\pi\sqrt{m}}$ to $\bar{\nu}_0 = \frac{n}{2\pi\sqrt{m}}\left(1 - \frac{k^2}{4mn^2}\right)^{\frac{1}{2}}$. For small values of k the damped frequency $\bar{\nu}_0$ is connected with the "natural" frequency by the relation

$$\bar{\nu}_0 = \nu_0\left(1 - \frac{k^2}{8mn^2}\right),$$

so that the effect of friction upon the frequency is of the second order in k.

(2) Here $k^2 > 4\,mn^2$. The two real roots of the characteristic equation are both negative, since m, k, and n^2 are all positive. Denoting these roots by $-\alpha$ and $-\beta$, the general solution of our differential equation is

$$x = Ae^{-\alpha t} + Be^{-\beta t},$$

where A and B are arbitrary constants. From this it follows that

$$\dot{x} = -A\alpha e^{-\alpha t} - B\beta e^{-\beta t},$$

so that $\dot{x}$ vanishes when $e^{(\alpha-\beta)t} + \frac{A\alpha}{B\beta} = 0$, and x itself vanishes when $e^{(\alpha-\beta)t} + \frac{A}{B} = 0$. If A and B have the same sign, neither of these equations is satisfied for any value of t; whereas if A and B have opposite signs, each is satisfied by a single value of t, provided that $-\frac{A}{B} > 1$, α being supposed to be greater than β. The motion is not periodic, and is termed aperiodic. The accompanying figures graph the single term $x = Ae^{-\alpha t}$ and the expression $x = e^{(\alpha-\beta)t}$.

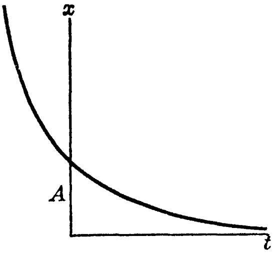


FIG. 21

$x = Ae^{-\alpha t}$, $\alpha > 0$. For $t > 0$, $x < A$

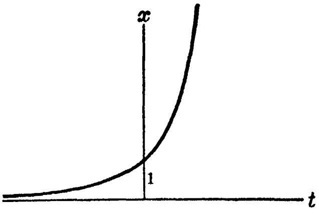


FIG. 22

$x = e^{(\alpha-\beta)t}$, $\alpha > \beta$. For $t > 0$, $x > 1$

If we write $\alpha = \lambda + \mu$, $\beta = \lambda - \mu$, where λ and μ are positive quantities, and if we measure t from the position $x = 0$, our solution takes the form $x = Ce^{-\lambda t}\sinh \mu t$. The maximum value of x then occurs at the time t given by

$$\tanh \mu t = \frac{\mu}{\lambda}.$$

(3) Here $k^2 = 4\,mn^2$. The general solution is

$$x = (At + B)e^{-\frac{kt}{2m}}$$

and the motion is again aperiodic. If A and B have opposite signs, x vanishes just once $\left(\text{when } t = -\frac{B}{A}\right)$, and $\dot{x}$ vanishes when $t = \frac{2m}{k} - \frac{B}{A}$.

In all three cases, (1), (2), and (3), the values of the constants of integration, for example, (C, δ) or (A, B), are to be determined by the initial conditions of motion. In other words, they are determined from the values of x and $\dot{x}$ at time $t = 0$.

39. Simple harmonic motion in space; Lissajous's figures. Let us consider the motion of a particle under the action of a central force which is directed toward the center O and whose magnitude is directly proportional to the displacement from this center. We have already seen, in § 37, that the motion takes place in a plane through O determined by the initial position and velocity of the particle. Taking a pair of rectangular axes OX and OY in this plane, we have, on resolving the acceleration and the force along these axes, the equations

$$m\ddot{x} = -n^2x, \quad m\ddot{y} = -n^2y;$$

whence

$$x = C_1 \cos(2\pi\nu_0 t - \delta_1),$$

$$y = C_2 \cos(2\pi\nu_0 t - \delta_2),$$

where $\nu_0 = \frac{n}{2\pi\sqrt{m}}$ and where (C_1, δ_1) and (C_2, δ_2) are constants of integration. These results are expressed by the statement that the motion is that due to the composition of two simple harmonic motions along two perpendicular lines, the frequencies of the two vibrations being the same, but the amplitudes and phases being in general different.

In order to determine the path of the particle, we eliminate t from the two equations above. Writing

$$\begin{aligned} y &= C_2 \cos[2\pi\nu_0 t - \delta_1 + (\delta_1 - \delta_2)] \\ &= \frac{C_2}{C_1} x \cos(\delta_1 - \delta_2) - C_2\sqrt{1 - \frac{x^2}{C_1^2}} \sin(\delta_1 - \delta_2), \end{aligned}$$

and using the abbreviation $\epsilon = \delta_1 - \delta_2$ for the difference in phase of the two vibrations, we find

$$(C_1y - C_2x\cos\epsilon)^2 = C_2^2(C_1^2 - x^2)\sin^2\epsilon,$$

or

$$C_2^2x^2 - 2C_1C_2\cos\epsilon\, xy + C_1^2y^2 = C_1^2C_2^2\sin^2\epsilon. \quad (39.1)$$

The path is therefore a conic, and since the second-degree terms in its equation have imaginary factors, it is an ellipse, the asymptotes being imaginary. (The type of conic furnished by a given second-degree equation is determined by the character of the equation for large values of x and y. For such values the second-degree terms dominate the situation. If they are broken into a product of two linear factors, these give us, when equated separately to zero, lines parallel to the asymptotes of the conic. Hence, if the two linear factors are real and distinct, the conic is a hyperbola; if they are imaginary, the conic is an ellipse; and finally, if they are not distinct, the conic is a parabola.) If, however, $\epsilon = 0$ (or π), the ellipse degenerates into part of one of the lines

$$C_2x - C_1y = 0 \quad (\text{or} \quad C_2x + C_1y = 0),$$

and the motion is again a simple harmonic rectilinear motion.

It is a simple exercise to show that when the amplitudes are equal, — that is, $C_1 = C_2 = C$ (say), — the axes of (39.1) are inclined at an angle of 45 degrees to the axes of reference, and that the lengths of the axes are the numeric values of $2\,C \cos\frac{\epsilon}{2}$ and $2\,C \sin\frac{\epsilon}{2}$.

In the general case the center of the ellipse (39.1) coincides with the center of attraction O, and the axes of the ellipse are inclined to the axes of reference, along which the component simple harmonic vibrations take place, at the angle $\frac{1}{2}\operatorname{arc\,tan}\frac{2\,C_1C_2\cos\epsilon}{C_1{}^2 - C_2{}^2}$. When $\epsilon = \pm\frac{\pi}{2}$, the axes of the ellipse coincide with the axes of reference, and its equation takes the simple form

$$\frac{x^2}{C_1{}^2} + \frac{y^2}{C_2{}^2} = 1. \tag{39.2}$$

We have a circular vibration when $C_1 = C_2$, that is, when the two simple harmonic vibrations along two perpendicular lines have the same amplitudes, their phases differing by a right angle. There are two types of such circular harmonic motions, corresponding to $\epsilon = \frac{\pi}{2}$ and $\epsilon = -\frac{\pi}{2}$ respectively.

Let us combine two circular vibrations, one of each type,

$$\begin{Bmatrix} x = C\cos(2\,\pi\nu_0 t - \delta_1) \\ y = C\sin(2\,\pi\nu_0 t - \delta_1) \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} x = C\cos(2\,\pi\nu_0 t - \delta_2) \\ y = -\,C\sin(2\,\pi\nu_0 t - \delta_2) \end{Bmatrix},$$

the amplitudes and frequencies being the same for each, but the phases being in general different. The first of the two circular vibrations is of the counterclockwise type (when viewed from a point on the positive half of the z-axis), whereas the second is of the clockwise type. We find

$$x = 2\,C \cos\frac{\delta_2 - \delta_1}{2}\cos\left(2\,\pi\nu_0 t - \frac{\delta_1 + \delta_2}{2}\right)$$

and
$$y = 2\,C \sin\frac{\delta_2 - \delta_1}{2}\cos\left(2\,\pi\nu_0 t - \frac{\delta_1 + \delta_2}{2}\right);$$

whence $\frac{y}{x} = \tan\frac{\delta_2 - \delta_1}{2}$, and we have a simple harmonic vibration

$$r = 2\,C \cos\left(2\,\pi\nu_0 t - \frac{\delta_1 + \delta_2}{2}\right)$$

along this straight line, which makes an angle $\frac{\delta_2 - \delta_1}{2}$ with the x-axis.

When the frequencies of the two component simple harmonic vibrations are different, the path of the particle is not, in general, an ellipse. Thus let us consider the two simple harmonic motions

$$x = C_1 \cos(2\,\pi\nu_1 t - \delta_1), \quad y = C_2 \cos(2\,\pi\nu_2 t - \delta_2),$$

where we shall suppose the two frequencies commensurable. We may therefore write $\nu_1 = \alpha p$, $\nu_2 = \alpha q$, where p and q are two integers which are mutually prime, that is, which have no common factor. The period of the first vibration being $\frac{1}{\nu_1} = \frac{1}{\alpha p}$, and that of the second being $\frac{1}{\nu_2} = \frac{1}{\alpha q}$, we see that after $\frac{1}{\alpha}$ units of time the first vibration has completed p swings, and the second q swings. Hence the curve traced out by the particle is a closed one, and the motion in it is periodic, the period being $\frac{1}{\alpha}$. Such a curve is known as a Lissajous figure, and its explicit equation follows from

$$\text{arc cos}\,\frac{x}{C_1} = 2\,\pi\nu_1 t - \delta_1 = 2\,\pi\alpha p t - \delta_1,$$

$$\text{arc cos}\,\frac{y}{C_2} = 2\,\pi\nu_2 t - \delta_2 = 2\,\pi\alpha q t - \delta_2,$$

and is
$$q\,\text{arc cos}\,\frac{x}{C_1} - p\,\text{arc cos}\,\frac{y}{C_2} = p\delta_2 - q\delta_1. \qquad (39.3)$$

To each pair of integers p and q, which are determined by the ratio $\nu_1 : \nu_2$, we have a single Lissajous curve, the amplitudes and phases of the component simple harmonic vibrations being supposed to be kept unaltered (the frequencies alone being changed, and these in such a manner that their ratio is unaltered). Furthermore, a change in the right-hand side of (39.3) by any multiple of 2π does not affect the coördinates x and y on the left, and so leads to the same Lissajous curve. For example, a change in δ_2 amounting to an integral multiple of $\frac{2\pi}{p}$, δ_1 being unaltered, gives the same Lissajous curve.

It is important to notice that when the two frequencies ν_1 and ν_2 are nearly equal, we may regard the corresponding Lissajous curve as to all intents and purposes a continuously changing ellipse. Thus, if $\nu_2 = \nu_1 + \eta$, we have

$$x = C_1 \cos(2\pi\nu_1 t - \delta_1), \quad y = C_2 \cos[2\pi\nu_1 t - (\delta_2 - 2\pi\eta t)];$$

and if we suppose η to be small in comparison with ν_1, $\delta_2 - 2\pi\eta t$ may be regarded as practically constant during the time $\frac{1}{\nu_1}$ of a single vibration, so that the part of the Lissajous curve traced out during such a vibration is practically an ellipse. Since the axes of the ellipse make an angle $\frac{1}{2}\arctan\frac{2C_1C_2\cos(\delta_1 - \delta_2 + 2\pi\eta t)}{C_1^2 - C_2^2}$ with the axes of coördinates, we see that the axes of the ellipse make a complete revolution in the period $\pm\frac{1}{\eta}$, so that the frequency of rotation of the axes is $\pm\eta = \pm(\nu_2 - \nu_1)$. If $C_1 > C_2$, the rotation is clockwise, since then the arc tan is decreasing and positive (at the instant $\delta_1 - \delta_2 + 2\pi\eta t = 0$); whereas if $C_1 < C_2$, the rotation is counterclockwise. In the intermediate case, $C_1 = C_2$, no rotation takes place, and the axes of the ellipse always bisect the angles between the axes of coördinates. The continuously changing phase difference $\delta_1 - (\delta_2 - 2\pi\eta t)$ causes a continuously changing shape of the ellipse. After a period $\pm\frac{1}{\eta}$ the original shape is regained.

In a similar manner we may deal with the case of two harmonic vibrations

$$x = C_1 \cos(2\pi\nu_1 t - \delta_1), \quad y = C_2 \cos(2\pi\nu_2 t - \delta_2),$$

where $\nu_1 = \alpha p$, $\nu_2 = \alpha q + \eta$, and p and q are integers which have no common factor. Writing $y = C_2 \cos[2\pi\alpha q t - (\delta_2 - 2\pi\eta t)]$, we may regard $\delta_2 - 2\pi\eta t$ as practically constant during the apparent period $\frac{1}{\alpha}$ of the Lissajous curve (η being supposed to be small in comparison with $q\alpha$). The equation of the Lissajous curve being $q \text{ arc cos } \frac{x}{C_1} - p \text{ arc cos } \frac{y}{C_2} = p\delta_2 - q\delta_1 - 2\pi p\eta t$, we recover the original curve after a lapse of time $\pm \frac{1}{p\eta}\cdot$ Hence the frequency of the Lissajous curve, $\pm p\eta$, is $\pm (p\nu_2 - q\nu_1)$; p and q are determined by the form of the revolving curve, and the frequency of the return of the figure may readily be determined by experiment. Hence, if ν_1 is known, ν_2 may be determined in this way. The sign to be used may readily be found by slightly changing ν_1 and observing whether the frequency of the revolving curve is thereby increased or decreased. The curve is not a closed one, although it may appear to the eye to be closed.

40. The forced vibrations of a simple harmonic oscillator. We shall now treat the problem of the rectilinear motion of a particle where, in addition to the elastic restoring force, and possibly the frictional, or damping, force, there is a force $\mathbf{F}(t)$ which does not depend on the position or state of motion of the particle and which acts on the particle in a direction along its line of motion. The equation governing the motion is of the form

$$m\ddot{x} + k\dot{x} + n^2x = f(t),$$

where the numeric value of $f(t)$ is the magnitude of $\mathbf{F}(t)$. In order to find the general solution of this nonhomogeneous, linear, differential equation of the second order, we have merely to find some one particular solution and to add to it the general solution of the homogeneous equation

$$m\ddot{x} + k\dot{x} + n^2x = 0,$$

which we have already treated. In proceeding to find the necessary particular solution, we first notice that if $f(t)$ be split up into two parts, $f_1(t)$ and $f_2(t)$, so that $f(t) = f_1(t) + f_2(t)$, and if x_1 and x_2 be particular solutions of the equations

$$m\ddot{x} + k\dot{x} + n^2x = f_1(t) \quad \text{and} \quad m\ddot{x} + k\dot{x} + n^2x = f_2(t)$$

respectively, then $x_1 + x_2$ is a particular solution of the equation

$$m\ddot{x} + k\dot{x} + n^2x = f_1(t) + f_2(t) = f(t),$$

which we have to solve. Since any function $f(t)$, of the type likely to occur in practice, may be, for any time interval in which we are interested, expanded in a Fourier series of sines and cosines of multiples of t, we first proceed to discuss the problem of determining a particular solution of the equation

$$m\ddot{x} + k\dot{x} + n^2x = A \cos 2\pi\nu t \tag{40.1}$$

corresponding to the case where the external applied force is simply harmonic, its amplitude being denoted by A and its frequency by ν. It will be convenient to consider in conjunction with (40.1) the auxiliary equation

$$m\ddot{x} + k\dot{x} + n^2x = A \sin 2\pi\nu t, \tag{40.2}$$

which corresponds to an external applied simple harmonic force of the same amplitude and frequency as before, but with a phase less by a right angle. If u is any solution of (40.1), and v any solution of (40.2), then the complex function $w = u + iv$ is a solution of the equation

$$m\ddot{x} + k\dot{x} + n^2x = Ae^{2\pi i\nu t}. \tag{40.3}$$

Conversely, if w is any solution of (40.3), its real part u is a solution of (40.1), and its imaginary part v is a solution of (40.2). Since (40.3) is easier to deal with than either (40.1) or (40.2), it is convenient first to show how to find a particular solution of (40.3). To do this, we introduce the new dependent variable z by means of the equation $w = ze^{2\pi i\nu t}$, and we find, upon substitution in (40.3), that z satisfies the equation

$$m\ddot{z} + (4\pi i\nu m + k)\dot{z} + (-4\pi^2\nu^2 m + 2\pi i\nu k + n^2)z = A.$$

On denoting the characteristic polynomial $m\alpha^2 + k\alpha + n^2$ by $f(\alpha)$, we have

$$\frac{f''(2\pi i\nu)}{1.2}\ddot{z} + f'(2\pi i\nu)\dot{z} + f(2\pi i\nu)z = A. \tag{40.4}$$

It follows that if $2\pi i\nu$ is *not* a zero of the characteristic polynomial $f(\alpha)$, we may choose for z the constant value

$$z = \frac{A}{n^2 - 4\pi^2 m\nu^2 + 2\pi i\nu k}, \tag{40.5}$$

and the solution of (40.3) is $ze^{2\pi i\nu t}$. If, on the other hand, $2\pi i\nu$ is a zero of the characteristic polynomial, we may take for z the linear function of t,

$$z = \frac{At}{k + 4\pi i\nu m}. \tag{40.5a}$$

Since k and m are real, there is no possibility for the denominator of this ratio to vanish.

In general, if we have a nonhomogeneous linear differential equation of any order with constant coefficients, and with right-hand side $Ae^{2\pi i\nu t}$, a particular solution is

$$w = \frac{At^p e^{2\pi i\nu t}}{f^{(p)}(2\pi i\nu)},$$

where $2\pi i\nu$ is a zero of multiplicity p of the characteristic polynomial $f(\alpha)$.

Let us consider first the case where $2\pi i\nu$ is not a zero of the characteristic polynomial $m\alpha^2 + k\alpha + n^2$. This will always be the situation unless $k = 0$. Since the modulus of z, as given by (40.5), is $\frac{A}{R}$, where

$$R = [(n^2 - 4\pi^2 m\nu^2)^2 + 4\pi^2 k^2\nu^2]^{\frac{1}{2}},$$

and since its argument is $-\delta$, where

$$\tan\delta = \frac{2\pi k\nu}{n^2 - 4\pi^2 m\nu^2},$$

the real part of $w = ze^{2\pi i\nu t}$ is

$$u = \frac{A}{R}\cos(2\pi\nu t - \delta). \tag{40.6}$$

This is the required particular solution of (40.1). It represents a simple harmonic vibration of the same frequency as the applied force, and is known as the forced vibration. The essential thing to notice about the forced vibration is that it is undamped, and that its frequency depends on that of the applied force alone and not at all upon the nature of the vibrating particle. When there is a frictional, or damping, force, and $k^2 < 4mn^2$, the free vibrations, which are given by the solutions of the homogeneous equation $m\ddot{x} + k\dot{x} + n^2x = 0$, are rapidly damped out, while the forced vibration persists as a pure harmonic. The seismograph, an apparatus for registering earthquakes, depends upon this fact. Here the frequency of the disturbing force is required, so that the damping factor is made large in order to remove immediately the transient free vibrations.

It may at first sight be surprising that the frequency of the forced vibration should be independent of the vibrator and that its amplitude should be undamped. The following analysis may tend to explain how this happens. The forced vibration being

$$x = \frac{A\cos(2\,\pi\nu t - \delta)}{[(n^2 - 4\,\pi^2 m\nu^2)^2 + 4\,\pi^2 k^2\nu^2]^{\frac{1}{2}}},$$

where $\tan\delta = \dfrac{2\,\pi k\nu}{n^2 - 4\,\pi^2 m\nu^2}$, it readily follows that

$$A\cos 2\,\pi\nu t = (n^2 - 4\,\pi^2 m\nu^2)x + k\dot{x}.$$

The part n^2x of the applied force $A\cos 2\,\pi\nu t$ goes to neutralize the elastic restoring force, and the part $k\dot{x}$ to neutralize the damping force. There remains the part $-4\,\pi^2 m\nu^2 x$ to yield a simple harmonic vibration of frequency ν.

a. The phenomenon of resonance. The amplitude of the forced vibration being

$$\frac{A}{[(n^2 - 4\,\pi^2 m\nu^2)^2 + 4\,\pi^2 k^2\nu^2]^{\frac{1}{2}}},$$

let us seek what frequency of the applied force is required in order that the amplitude may be a maximum for a given vibrator. Putting, for the moment, $4\,\pi^2\nu^2 = \xi$, we see that the amplitude is a maximum when the quadratic expression in ξ,

$$y = (n^2 - m\xi)^2 + k^2\xi = m^2\xi^2 + (k^2 - 2\,mn^2)\xi + n^4,$$

is a minimum. If we plot y against ξ on an ordinary Cartesian diagram, the graph is a parabola with its axis parallel to the y-axis and with its vertex at the point

$$\xi_0 = \frac{n^2}{m} - \frac{k^2}{2\,m^2}, \quad y_0 = k^2\left(\frac{n^2}{m} - \frac{k^2}{4\,m^2}\right).$$

This value of y furnishes the minimum ordinate of the parabola.

Since ξ is, by its definition, restricted to positive values, it follows that, if $\dfrac{n^2}{m} \leqq \dfrac{k^2}{2\,m^2}$, the minimum ordinate of the parabola, in the range to which we are restricted, occurs for $\xi = 0$ (that is, $\nu = 0$), the corresponding maximum amplitude being $\dfrac{A}{n^2}$. As the frequency ν increases, the amplitude of the forced vibration steadily decreases, being always less than $\dfrac{A}{n^2}$. Introducing the free undamped frequency ν_0 $\left(\text{namely, } \nu_0 = \dfrac{n}{2\,\pi\sqrt{m}}\right)$, the condition $\dfrac{n^2}{m} \leqq \dfrac{k^2}{2\,m^2}$ takes the form $\nu_0{}^2 \leqq \dfrac{k^2}{8\,\pi^2 m^2}$. There is no resonance effect in this case.

If, on the other hand, $\nu_0^2 > \dfrac{k^2}{8\,\pi^2 m^2}$, the amplitude of the forced vibration reaches its maximum when $\xi = \dfrac{n^2}{m} - \dfrac{k^2}{2\,m^2}$, that is, when

$$\nu^2 = \nu_0^2 - \frac{k^2}{8\,\pi^2 m^2} = \nu_0^2\left(1 - \frac{k^2}{2\,mn^2}\right). \tag{40.7}$$

This frequency ν is not only less than ν_0 but is also slightly less than the free damped frequency $\bar{\nu}_0$ which is given by the equation (38.5),

$$\bar{\nu}_0^2 = \nu_0^2 - \frac{k^2}{16\,\pi^2 m^2} = \nu_0^2\left(1 - \frac{k^2}{4\,mn^2}\right).$$

The maximum amplitude, corresponding to the critical frequency ν given by (40.7), is

$$\frac{A}{k\sqrt{\dfrac{n^2}{m} - \dfrac{k^2}{4\,m^2}}} = \frac{A}{2\,\pi k\bar{\nu}_0}, \tag{40.8}$$

and is large if k is small. The applied force is then said to be *in resonance* with the vibrator.

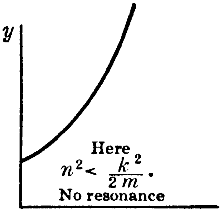


FIG. 23

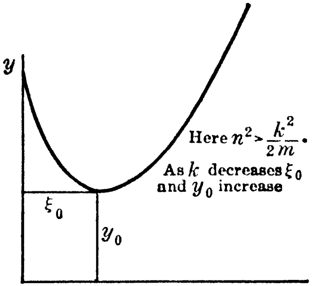


FIG. 24

If "sharp" resonance is desired, that is, if the amplitude of the forced vibration is to be negligible save for a small range of frequency near the resonance frequency given by (40.7), the parabola

$$y = m^2\xi^2 + (k^2 - 2\,mn^2)\xi + n^4,$$

or

$$y - y_0 = m^2(\xi - \xi_0)^2,$$

must be such that y is large compared with y_0 for fairly small values of $(\xi - \xi_0)$. This shows that y_0 must be small; or, in other words, the difference $4\,mn^2 - k^2$ must approach zero. The different cases are illustrated in the figures above.

We have seen that the phase of the forced vibration is $(2\,\pi\nu t - \delta)$, where $\tan\delta = \dfrac{2\,\pi k\nu}{n^2 - 4\,\pi^2 m\nu^2} = \dfrac{k\nu}{2\,\pi m(\nu_0{}^2 - \nu^2)}\cdot$ We see that when $\nu = \nu_0$, $\delta = \dfrac{\pi}{2}$; and, according as ν is greater or less than ν_0, δ is greater or less than $\dfrac{\pi}{2}\cdot$ The accompanying figure gives an idea of the variation of $\tan\delta$ with the frequency. We have already noted that the condition that there should be no resonance at all is $k^2 \geqq 2\,mn^2$. This condition is desirable in certain forms of seismographs. It is, however, required in these instruments that the free vibrations be transient, so that we must have $k^2 < 4\,mn^2$. The range of adjustment

$$2\,mn^2 \leqq k^2 < 4\,mn^2$$

is, accordingly, not very large.

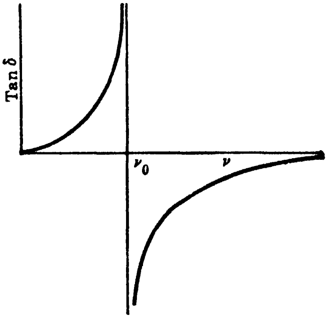


FIG. 25

The question of resonance may be looked at from another point of view. Thus, we may regard the frequency of the applied force as given, and ask for the vibrator which will yield a forced vibration of maximum amplitude. Assuming that k is the same for all vibrators, it is seen at once that $(n^2 - 4\,\pi^2 m\nu^2)^2$ must be a minimum; and this minimum value is zero, since a square cannot be negative. We have, then,

$$n^2 - 4\,\pi^2 m\nu^2 = 0 \quad \text{or} \quad \nu_0{}^2 = \nu^2.$$

The amplitude of this maximum forced vibration is $\dfrac{A}{2\,\pi k\nu}\cdot$

b. The energy of the forced vibration. The kinetic energy of the forced vibration is given by

$$T = \frac{1}{2}\,mv^2 = \frac{2\,A^2\pi^2\nu^2\sin^2(2\,\pi\nu t - \delta)}{[(n^2 - 4\,\pi^2 m\nu^2)^2 + 4\,\pi^2 k^2\nu^2]}. \tag{40.9}$$

The average value of this over a complete period is found by integrating it with respect to t between the limits t and $t + \dfrac{1}{\nu}$ and dividing the result by $\dfrac{1}{\nu}\cdot$ We find for this average value the result $\dfrac{A^2\pi^2\nu^2}{(n^2 - 4\pi^2 m\nu^2)^2 + 4\pi^2 k^2\nu^2}$. Considered as a function of ν^2 this has a maximum value when its reciprocal has a minimum value, that is, when $\dfrac{n^4}{\nu^2} + 16\pi^4 m^2\nu^2$ is a minimum. This occurs when $\nu^2 = \dfrac{n^2}{4\pi^2 m}$ or $\nu = \nu_0$. Thus, although the amplitude of the forced vibration is a maximum when $\nu = \nu_0\left(1 - \dfrac{k^2}{2\,mn^2}\right)^{\frac{1}{2}}$, the mean kinetic energy of the forced vibration is a maximum when $\nu = \nu_0$.

c. The forced vibrations of an undamped harmonic oscillator. Most of the results for the undamped vibrator follow from those for the damped vibrator by merely putting $k = 0$. The characteristic polynomial being $m\alpha^2 + n^2 = 0$, $2\pi i\nu$ is not a zero of it unless $\nu = \nu_0$. If $\nu \neq \nu_0$, we find $\delta = 0$ or π, so that the forced vibration has the same phase as (or else a phase differing by two right angles from) that of the exciting force. The forced vibration is given by

$$x = \frac{A\cos 2\pi\nu t}{n^2 - 4\pi^2 m\nu^2} = \frac{A\cos 2\pi\nu t}{4\pi^2 m(\nu_0^2 - \nu^2)}.$$

The amplitude of the vibration is the positive value of

$$\frac{A}{4\pi^2 m(\nu_0^2 - \nu^2)},$$

and it increases rapidly as ν approaches ν_0. When $\nu = \nu_0$, the formula for the forced vibration is, however, different. Since $2\pi i\nu$ is now a zero of the characteristic polynomial, x is the real part of $\dfrac{Ate^{2\pi i\nu t}}{4\pi i\nu m}$; that is,

$$x = \frac{At}{4\pi\nu m}\cos\left(2\pi\nu t - \frac{\pi}{2}\right) = \frac{At\sin 2\pi\nu t}{4\pi\nu m}. \qquad (40.10)$$

The phase of the forced vibration is accordingly behind that of the exciting force by a right angle, and the amplitude increases directly with the time. The whole argument ceases, of course, to have any application once the amplitude reaches the limit imposed by the hypothesis that the oscillations are small.

BIBLIOGRAPHY

Of the treatises mentioned on page 103, the books by Routh may be recommended for many illustrative examples and exercises. A classical reference is

RAYLEIGH. Theory of Sound, Vol. I. London, 1877.

A very complete treatment of the oscillation problem with practical applications is given in

HORT, W. Technische Schwingungslehre. Berlin, 1922.

EXERCISES

1. A particle of mass m is attached to the middle point of an elastic thread of natural length a and modulus λ, which is stretched between two fixed points. Prove that if no forces act on the particle other than the tensions in the parts of the thread, it can oscillate in the line of the thread with a simple harmonic motion of period $\pi\left(\frac{ma}{\lambda}\right)^{\frac{1}{2}}$.

2. A particle moves in a smooth tube in the form of a catenary, being attracted to the directrix with a force proportional to the distance from the directrix. Prove that the period of the oscillation is independent of the amplitude.

3. A particle moves along the axis of x, starting from rest at $x = a$. For an interval t_1 from the beginning of motion the acceleration is $-\mu x$. For a subsequent interval t_2 the acceleration is μx, and at the end of this interval the particle is at the origin. Prove that $\tan\sqrt{\mu}t_1 \tanh\sqrt{\mu}t_2 = 1$.

4. A particle moves in a straight line, tending to a fixed point in the line, under a force which, at distance r, is equal to $\frac{\mu}{r^2} - \frac{b^2\mu}{r^3a}$; and the particle starts from rest at distance $a + (a^2 - b^2)^{\frac{1}{2}}$. Prove that it will come to rest at distance $a - (a^2 - b^2)^{\frac{1}{2}}$ in time $\pi\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}$, and that it will oscillate between these points.

5. A shaft is imagined to be passed through the center of the earth. Discuss the motion of a body dropped into this shaft from the surface of the earth. The acceleration of the particle due to the attraction of the earth may be assumed to be proportional to the distance of the particle from the center of the earth.

6. What velocity is it necessary to give to a body, projected vertically upward from the surface of the earth, so that the body may never return?

7. Discuss the motion of a particle in a straight line under the action of a force always directed toward a fixed point on the line and proportional in magnitude to the distance from this point, and, in addition, of a constant frictional force whose direction is always opposite to the direction of motion.

8. A heavy particle P is suspended at rest from a point A by an elastic string whose initial and unstretched length is a. The point A, at the time $t = 0$, begins to oscillate up and down, so that its downward displacement at the time t is $c \sin \lambda t$. Prove that the length of the string at time t is

$$a + \frac{g}{n^2}(1 - \cos nt) - \frac{cn\lambda}{(n^2 - \lambda^2)} \sin nt + \frac{cn^2}{(n^2 - \lambda^2)} \sin \lambda t,$$

where $n = \frac{ma}{E}$, m being the mass of the particle and E being Young's modulus for the string.

9. A particle is attracted to several centers of force in a straight line, the magnitude of each force being proportional to the distance from the corresponding center. Show that the motion of the particle is simply harmonic.

10. Prove that in simple harmonic motion if the initial displacement be x_0 and the initial velocity u_0 the amplitude will be $\sqrt{\left(x_0^2 + \frac{u_0^2}{n^2}\right)}$ and the initial phase $-\tan^{-1} \frac{u_0}{nx_0}$.

11. A horizontal shelf is moved up and down with a simple harmonic motion of period T seconds. What is the greatest amplitude admissible so that a weight placed on the shelf may not be jerked off?

CHAPTER V

PROBLEMS OF NONRECTILINEAR MOTION OF A PARTICLE

41. The energy integral. The fundamental equation governing the motion of a particle is

$$m\mathbf{A} = \mathbf{F}, \tag{41.1}$$

where $\mathbf{A} = \ddot{\mathbf{r}}$ is the acceleration of the particle of mass m relative to a given reference frame, and $\mathbf{F}$ is the relative force acting on the particle, that is, the sum of the absolute and centrifugal forces if the reference frame is a noninertial one. Upon multiplying (41.1) scalarly by $\mathbf{v} = \dot{\mathbf{r}}$ and integrating with respect to t, we find

$$\frac{m}{2}(\dot{\mathbf{r}}.\dot{\mathbf{r}}) = \int_{t_0}^{t}(\mathbf{F}.\dot{\mathbf{r}})\,dt + \text{constant},$$

or

$$\tfrac{1}{2}mv^2 = \int_{t_0}^{t}(\mathbf{F}.\dot{\mathbf{r}})\,dt + \text{constant}.$$

If the integral $\int_{t_0}^{t}(\mathbf{F}.\dot{\mathbf{r}})\,dt$ on the right-hand side of these equations is positive, it is known as the work done by the force $\mathbf{F}$ on the mass particle in its motion during the interval t_0 to t; if negative, it is known as the work done against the force. If v_0 denotes the value of v when $t = t_0$, we have

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2$$
$$= \text{work done on particle by forces acting upon it.}$$

The quantity $\tfrac{1}{2}mv^2$ is called the kinetic energy of the particle relative to the reference frame in use and is usually denoted by T. We have, then, the result

$$\text{Increase in kinetic energy} = \text{work done by force system.} \tag{41.2}$$

The integrand $(\mathbf{F}.\dot{\mathbf{r}})$ of the work integral is known as the power, or activity, or rate of doing work of the force $\mathbf{F}$. We may, then, state (41.2) in the differential form

$$\text{Time rate of change of kinetic energy} = \text{power of applied force} \tag{41.3}$$

When the power $(\mathbf{F}.\dot{\mathbf{r}})$ is the time derivative of a function of x, y, z, and t, the result (41.2) gives the energy integral. Writing $(\mathbf{F}.\dot{\mathbf{r}}) = -\dot{V}$, we have

$$T + V = h, \tag{41.4}$$

where h is a constant known as the energy constant. V is known as the potential energy of the particle in the field of force $\mathbf{F}$, and is undefined to the extent of an additive constant. The reason for giving the name "potential energy" to the function V is that it is convenient to have a name for the various terms in the first integral (41.4) of the second-order differential equation (41.1). When there is no force acting on the particle, the first integral is $T = h$; and T has long been called the kinetic energy of the particle. If $\mathbf{F}$ is not such that $(\mathbf{F}.\dot{\mathbf{r}}) = -\dot{V}$, but is such that $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where $(\mathbf{F}_1.\dot{\mathbf{r}}) = -\dot{V}$, our equation (41.4) would have to be replaced by

$$\frac{d}{dt}(T + V) = (\mathbf{F}_2.\dot{\mathbf{r}}). \tag{41.5}$$

This says, after integration with respect to t, that the increase in $T + V$ is equal to the work done on the particle by $\mathbf{F}_2$, or that the decrease in $T + V$ is equal to the work expended by the particle against $\mathbf{F}_2$. When $\mathbf{F}_2$ is a frictional force, this energy appears as heat; and an extension of mechanical theory by which heat phenomena are regarded as mechanical in nature enables us to obtain again an energy integral. The idea of energy is therefore an evolutionary one.

It is important to notice that the work integral

$$\int_{t_0}^{t} (\mathbf{F}.\dot{\mathbf{r}})dt = -\int_{t_0}^{t} \dot{V}\,dt = V_0 - V,$$

so that the work done on the particle is equal to the negative change in potential energy of the particle in the field. A particularly important case occurs when V is a function of (x, y, z) only, and not explicitly of t. Then the value of the work integral depends merely on the initial and final positions of the particle and not at all on the path traversed by the particle between the positions. The force system is then said to be conservative. Thus a conservative field of force may be characterized by the fact that the work done by it on a particle which traverses any arbitrary closed path is zero. If, in addition, we make the assumption that $\mathbf{F}$ is a function of (x, y, z) alone, and not explicitly of t or the velocity components $(\dot{x}, \dot{y}, \dot{z})$ of the particle, we have, since the work integral

$$\int_{P_0}^{P} (\mathbf{F}.d\mathbf{r}) = V_0 - V$$

is independent of the path connecting P_0 and P (that is, is the integral of an exact differential),

$$\mathbf{F} = -\operatorname{grad} V. \qquad \text{[See § 13, } a\text{]}$$

It should be observed, however, that while this is sufficient for the existence of an energy integral, it is not a necessary condition if we allow $\mathbf{F}$ to depend on the velocity of the particle. Thus $\mathbf{F}$ might contain terms, such as those known as the Coriolis centrifugal-force terms, which are always perpendicular to $\dot{\mathbf{r}}$. These do not contribute anything to the work integral.

42. The angular-momentum integral. If the field of force $\mathbf{F}$ is such that, for every position of the mass particle m, the localized vector $\mathbf{F}$ is coplanar with a fixed straight line, we may at once obtain an integral of the fundamental equation (41.1). For, upon multiplying this equation vectorially by $\mathbf{r}$, we obtain

$$m[\mathbf{r}.\ddot{\mathbf{r}}] = [\mathbf{r}.\mathbf{F}],$$

or

$$\frac{d}{dt}[\mathbf{r}.m\dot{\mathbf{r}}] = [\mathbf{r}.\mathbf{F}]. \tag{42.1}$$

We may now regard $m\dot{\mathbf{r}}$ and $\mathbf{F}$ as localized vectors whose axes pass through the position of the mass particle m. The vector $m\dot{\mathbf{r}}$ is known as the linear momentum of the particle. The vector products $[\mathbf{r}.m\dot{\mathbf{r}}]$ and $[\mathbf{r}.\mathbf{F}]$ are, then, the vector moments about the origin of the linear-momentum vector and the force vector respectively, and the equation (42.1) may be stated as follows:

The time rate of change of angular momentum about any fixed point is equal to the moment of the applied force about this point. (The term "angular momentum" is here used to mean the vector moment of linear momentum.)

When $\mathbf{F}$ is coplanar with a fixed line, we take our fixed point on this line so that $[\mathbf{r}.\mathbf{F}]$ is perpendicular to the given fixed line. Hence the resolved part of the vector $\frac{d}{dt}[\mathbf{r}.m\dot{\mathbf{r}}]$ along the fixed line is zero. We may write this in the form

$$\left(\mathbf{A}_1 \cdot \frac{d}{dt}[\mathbf{r}.m\dot{\mathbf{r}}]\right) = 0,$$

where $\mathbf{A}_1$ is the unit vector in either direction along the fixed straight line. Since $\mathbf{A}_1$ is constant, this is equivalent to

$$\frac{d}{dt}(\mathbf{A}_1.\mathbf{r}.m\dot{\mathbf{r}}) = 0$$

or

$$(\mathbf{A}_1.\mathbf{r}.m\dot{\mathbf{r}}) = C,$$

where C is a constant. This is the first integral mentioned. The result just obtained may be stated as follows:

When the applied force **F** *is always coplanar with a fixed straight line, the resolved part of the angular momentum of the particle about any point of the line along this straight line is constant.*

If **F** always passes through a fixed point, this result holds for every line through this point, and we have the following result:

When the localized vector **F** *always has its axis passing through a fixed point, the angular momentum of the particle about this fixed point is constant.*

This result is an immediate consequence of (42.1); for if the fixed point is taken as our origin, then, by hypothesis, $[\mathbf{r}.\mathbf{F}] = \mathbf{0}$, so that $[\mathbf{r}.m\dot{\mathbf{r}}] = \mathbf{C}$, a constant vector.

43. Motion of a particle on a curve or surface or in space. Lagrange's equations. *a. Motion on a curve.* If a particle m is moving, under the action of a force **F**, on a curve whose equation is of the form

$$\mathbf{r} = \mathbf{r}(q), \qquad (43.1)$$

where q is any convenient parameter or independent variable, it will be useful to equate the resolved parts of the two equal vectors $m\ddot{\mathbf{r}}$ and **F** along the tangent to this curve. In forming this equation we may neglect any terms in **F** which are perpendicular to the tangent, that is, any terms in **F** whose activity, or rate of doing work on the particle, is zero. Thus, if the curve is a material one along which the particle is forced to move, there will be in general, in addition to the known external applied force, a force which is exerted by the curve on the particle, and which is known as the force of *reaction* of the curve. It is frequently possible to assume, at any rate as a first approximation, that the force of reaction is everywhere normal to the curve. Hence this reaction need not be considered in forming the equation we propose to deduce. In this case the motion is said to be frictionless, and the curve is called smooth.

If ds denotes the arc distance, measured from any convenient origin, along the curve (43.1), we have, say,

$$\frac{ds}{dq} = \text{magnitude of } \frac{d\mathbf{r}}{dq} = \sqrt{A},$$

so that

$$ds = \sqrt{A}\,dq,$$

where

$$A = \left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2;$$

and we may write

$$\frac{d\mathbf{r}}{dq} = \sqrt{A}\mathbf{t},$$

where $\mathbf{t}$ is the unit vector along the tangent drawn in the direction of motion (see § 13, *a*).

The resolved part of $\mathbf{F}$ along the tangent is, therefore,

$$(\mathbf{F}.\mathbf{t}) = \frac{1}{\sqrt{A}}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dq}\right);$$

and we shall write this in the form $\dfrac{Q}{\sqrt{A}}$, where

$$Q = \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dq}\right) = X\frac{dx}{dq} + Y\frac{dy}{dq} + Z\frac{dz}{dq}, \tag{43.2}$$

X, Y, Z being the components of $\mathbf{F}$.

The resolved part of $m\ddot{\mathbf{r}}$ along the tangent is, similarly,

$$\frac{1}{\sqrt{A}}\left(m\ddot{\mathbf{r}}\cdot\frac{d\mathbf{r}}{dq}\right) = \frac{1}{\sqrt{A}}\left[\frac{d}{dt}\left(m\dot{\mathbf{r}}\cdot\frac{d\mathbf{r}}{dq}\right) - m\dot{\mathbf{r}}\cdot\frac{d}{dt}\left(\frac{d\mathbf{r}}{dq}\right)\right].$$

But

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dq}\cdot\dot{q},$$

and hence

$$\frac{\partial\dot{\mathbf{r}}}{\partial\dot{q}} = \frac{d\mathbf{r}}{dq} \quad\text{and}\quad \frac{\partial\dot{\mathbf{r}}}{\partial q} = \frac{d^2\mathbf{r}}{dq^2}\cdot\dot{q} = \frac{d}{dt}\,\frac{d\mathbf{r}}{dq};$$

so that we may write, for the resolved part of $m\ddot{\mathbf{r}}$ along the tangent, the expression

$$\frac{1}{\sqrt{A}}\left[\frac{d}{dt}\left(m\dot{\mathbf{r}}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial\dot{q}}\right) - \left(m\dot{\mathbf{r}}\cdot\frac{\partial}{\partial q}\dot{\mathbf{r}}\right)\right] = \frac{1}{\sqrt{A}}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}}\right) - \frac{\partial T}{\partial q}\right], \tag{43.3}$$

where $T = \frac{1}{2}\,m(\dot{\mathbf{r}}.\dot{\mathbf{r}}) = \frac{1}{2}\,mv^2$ is the kinetic energy of the particle. Hence we have the result

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}} - \frac{\partial T}{\partial q} = Q. \tag{43.4}$$

The resolved part of the acceleration vector $\ddot{\mathbf{r}}$ along the tangent is found (by considering the motion of a particle of unit mass) to be

$$\frac{1}{\sqrt{A}}\left(\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}} - \frac{\partial T_1}{\partial q}\right), \quad \text{where} \quad T_1 = \frac{1}{2}v^2. \tag{43.5}$$

It should be observed that the term $\dfrac{\partial T}{\partial \dot{q}} = \left(m\dot{\mathbf{r}} \cdot \dfrac{d\mathbf{r}}{dq}\right)$ is equal to the product of $\sqrt{A}$ by the resolved part of the linear-momentum vector along the tangent. The resolved part of the velocity vector along the tangent or, in fact, the magnitude of the velocity vector is found (by considering again a particle of unit mass) to be

$$\frac{1}{\sqrt{A}}\frac{\partial T_1}{\partial \dot{q}}, \quad \text{where} \quad T_1 = \frac{1}{2}v^2. \tag{43.6}$$

NOTE. The curve (43.1) is a fixed curve in the reference frame in use. If we wish to consider a particle moving on a curve which is itself in motion, the equation (43.1) must be replaced by one of the type

$$\mathbf{r} = \mathbf{r}(q, t). \tag{43.1 a}$$

The time t now occurs explicitly in the equation of the curve. At any instant t we have $ds = \sqrt{A}\,dq$ and $\dfrac{\partial \mathbf{r}}{\partial q} = \sqrt{A}\,\mathbf{t}$, where $A = \left(\dfrac{\partial x}{\partial q}\right)^2 + \left(\dfrac{\partial y}{\partial q}\right)^2 + \left(\dfrac{\partial z}{\partial q}\right)^2$ is the square of the magnitude of $\dfrac{\partial \mathbf{r}}{\partial q}$ at the instant in question. The same results as above follow. The only modification in the argument is that

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial q} \cdot \dot{q} + \frac{\partial \mathbf{r}}{\partial t};$$

whence

$$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}} = \frac{\partial \mathbf{r}}{\partial q}$$

and

$$\frac{\partial \dot{\mathbf{r}}}{\partial q} = \frac{\partial^2 \mathbf{r}}{\partial q^2} \cdot \dot{q} + \frac{\partial^2 \mathbf{r}}{\partial q\, \partial t} = \frac{d}{dt}\frac{\partial \mathbf{r}}{\partial q}.$$

It is important to notice that the forces which may now be neglected in forming Q are those whose scalar products into $\dfrac{\partial \mathbf{r}}{\partial q}$ are zero, that is, those which are normal to the curve at the instant t in question. Since $\dot{\mathbf{r}} = \dfrac{\partial \mathbf{r}}{\partial q} \cdot \dot{q} + \dfrac{\partial \mathbf{r}}{\partial t}$, the activity of these forces is not now zero, in general, since its value is their scalar product into $\dfrac{\partial \mathbf{r}}{\partial t}$. The integrand of the work integral is $\left(\mathbf{F} \cdot \dfrac{\partial \mathbf{r}}{\partial q}\right)dq + \left(\mathbf{F} \cdot \dfrac{\partial \mathbf{r}}{\partial t}\right)dt$, and it is usual to call the first term of this integrand, that is, $\left(\mathbf{F} \cdot \dfrac{\partial \mathbf{r}}{\partial q}\right)dq$, the *virtual work* of the force **F** acting on the particle. We may say, then, that the forces which may be neglected in forming the equation (43.2) are those *whose virtual work is zero.*

b. Motion on a surface. The argument here is precisely similar to the foregoing discussion and may be given briefly. To take care of the situation where the surface is moving, we write its equation in the form

$$\mathbf{r} = \mathbf{r}(q_1, q_2, t) \tag{43.7}$$

where q_1 and q_2 are any two convenient parameters or independent variables. On putting, in turn, $q_2 =$ constant and $q_1 =$ constant, we obtain, for any value of t, two sets of curves on the surface, each set being found by varying the constant value assigned to the corresponding parameter q. These curves are known as the parametric curves of the surface. At any instant we have, at any point of the surface, two unit vectors tangent to the two parametric curves through the point. They are given by the formulas

$$\frac{\partial \mathbf{r}}{\partial q_1} = \sqrt{A_1}\,\mathbf{t}_1, \quad \frac{\partial \mathbf{r}}{\partial q_2} = \sqrt{A_2}\,\mathbf{t}_2,$$

where

$$A_1 = \left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2,$$

and

$$A_2 = \left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial z}{\partial q_2}\right)^2.$$

Upon resolving **F** and $m\ddot{\mathbf{r}}$ along the tangent to the parametric curve $q_2 =$ constant, we obtain, exactly as before, the equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1, \tag{43.8}$$

where

$$Q_1 = \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_1}\right) = X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} + Z\frac{\partial z}{\partial q_1},$$

and $T = \frac{1}{2}mv^2$ is the kinetic energy of the moving particle. Similarly, on resolving along the tangent to the parametric line $q_1 =$ constant, we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = Q_2, \tag{43.9}$$

where

$$Q_2 = \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_2}\right).$$

Since $\dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial q_1} \cdot \dot{q}_1 + \frac{\partial \mathbf{r}}{\partial q_2} \cdot \dot{q}_2 + \frac{\partial \mathbf{r}}{\partial t}$, we obtain for T the expression

$$2\,T = m(A_1\dot{q}_1{}^2 + 2\,B\dot{q}_1\dot{q}_2 + A_2\dot{q}_2{}^2 + 2\,F\dot{q}_1 + 2\,G\dot{q}_2 + H),$$

where A_1 and A_2 have the values given above, and

$$B = \left(\frac{\partial \mathbf{r}}{\partial q_1} \cdot \frac{\partial \mathbf{r}}{\partial q_2}\right), \quad F = \left(\frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial q_1}\right),$$

$$G = \left(\frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial q_2}\right), \quad H = \left(\frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial t}\right).$$

When the surface is fixed, F, G, and H are all zero, and we have for T the simpler form

$$2\,T = m(A_1\dot{q}_1{}^2 + 2\,B\dot{q}_1\dot{q}_2 + A_2\dot{q}_2{}^2).$$

It will be observed that

$$B = \sqrt{A_1A_2}(\mathbf{t}_1 . \mathbf{t}_2) = \sqrt{A_1A_2}\cos\alpha,$$

where α is the angle between the two parametric lines through the point in question. The resolved parts of the acceleration vector along the two parametric lines are, respectively,

$$\frac{1}{\sqrt{A_1}}\left(\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_1} - \frac{\partial T_1}{\partial q_1}\right) \quad \text{and} \quad \frac{1}{\sqrt{A_2}}\left(\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_2} - \frac{\partial T_1}{\partial q_2}\right),$$

where T_1 is the kinetic energy of a particle of unit mass.

It will be observed that any forces which are normal to the surface at the instant t may be neglected in forming the expressions Q_1 and Q_2. The activity of any force **F** acting on the particle is

$$(\mathbf{F} . \dot{\mathbf{r}}) = \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_1}\right)\dot{q}_1 + \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_2}\right)\dot{q}_2 + \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial t}\right).$$

The integrand of the work integral is

$$\left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_1}\right)dq_1 + \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_2}\right)dq_2 + \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial t}\right).$$

The first two terms of this expression constitute what is known as the *virtual work* of the force **F**; and we may neglect, in forming the equations (43.8) and (43.9), any forces whose virtual work is zero for all values of dq_1 and dq_2, or, in other words, for an arbitrary *virtual displacement.* A virtual displacement defines a vector in the plane tangent to the surface for a given value of t, whereas the *actual displacement* of the particle defines the direction of motion and, for a moving surface, is not, in general, identical with any virtual displacement. Thus, if we write

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q_1} dq_1 + \frac{\partial \mathbf{r}}{\partial q_2} dq_2 + \frac{\partial \mathbf{r}}{\partial t} dt,$$

the first two terms constitute a virtual displacement, which we may denote by δr; and we have

$$d\mathbf{r} = \delta \mathbf{r} + \frac{\partial \mathbf{r}}{\partial t} dt. \tag{43.10}$$

For a fixed surface $\frac{\partial \mathbf{r}}{\partial t} = 0$, and the actual displacement dr coincides with a virtual displacement $\delta \mathbf{r}$.

If the force of reaction or constraint exerted by a material surface upon the particle moving on it is such that its virtual work is zero, the surface is said to be smooth, and the motion is said to be frictionless. The reaction in this case, at any instant, is normal to the surface at that instant. In discussing the motion of a particle upon a smooth surface the force of reaction may be neglected in forming the equations (43.8) and (43.9).

c. Motion in space in terms of curvilinear coördinates. Here the theory is the same as in the two foregoing cases, and it will not be necessary to do more than indicate the argument. We have a system of curvilinear coördinates (q_1, q_2, q_3), so that

$$\mathbf{r} = \mathbf{r}(q_1, q_2, q_3);$$

whence

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{r}}{\partial q_2} \dot{q}_2 + \frac{\partial \mathbf{r}}{\partial q_3} \dot{q}_3,$$

giving

$$T = \tfrac{1}{2} m(\dot{\mathbf{r}}.\dot{\mathbf{r}}) = \frac{m}{2} (A_1\dot{q}_1^2 + A_2\dot{q}_2^2 + A_3\dot{q}_3^2 + 2\,B_1\dot{q}_2\dot{q}_3 + 2\,B_2\dot{q}_3\dot{q}_1 + 2\,B_3\dot{q}_1\dot{q}_2),$$

where

$$A_1 = \left(\frac{\partial \mathbf{r}}{\partial q_1} \cdot \frac{\partial \mathbf{r}}{\partial q_1}\right) = \left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2,$$

with similar expressions for A_2 and A_3, and

$$B_1 = \left(\frac{\partial \mathbf{r}}{\partial q_2} \cdot \frac{\partial \mathbf{r}}{\partial q_3}\right) = \frac{\partial x}{\partial q_2}\frac{\partial x}{\partial q_3} + \frac{\partial y}{\partial q_2}\frac{\partial y}{\partial q_3} + \frac{\partial z}{\partial q_2}\frac{\partial z}{\partial q_3},$$

with similar expressions for B_2 and B_3. Denoting by $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, respectively, the three unit vectors tangent at any point to the three coördinate lines (q_2 and q_3 constants etc.) through that point, we have

$$\frac{\partial \mathbf{r}}{\partial q_1} = \sqrt{A_1}\mathbf{t}_1, \quad \frac{\partial \mathbf{r}}{\partial q_2} = \sqrt{A_2}\mathbf{t}_2, \quad \frac{\partial \mathbf{r}}{\partial q_3} = \sqrt{A_3}\mathbf{t}_3,$$

so that $B_1 = \sqrt{A_2A_3}(\mathbf{t}_2 . \mathbf{t}_3) = \sqrt{A_2A_3}\cos\alpha,$

where α is the angle between the second and third coördinate lines, and so on. Upon resolving both the vectors $m\ddot{\mathbf{r}}$ and $\mathbf{F}$ along the three coördinate lines in turn, we find

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} &= Q_1, \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} &= Q_2, \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_3} - \frac{\partial T}{\partial q_3} &= Q_3, \end{aligned} \tag{43.11}$$

where $Q_1 = \left(\mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial q_1}\right), \quad Q_2 = \left(\mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial q_2}\right), \quad Q_3 = \left(\mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial q_3}\right).$

Thus the integrand of the work integral is $Q_1dq_1 + Q_2dq_2 + Q_3dq_3$. For this reason the expressions Q are known as *generalized forces.* The resolved part of the linear-momentum vector of the particle along the first coördinate line is

$$\frac{1}{\sqrt{A_1}}\frac{\partial T}{\partial \dot{q}_1},$$

and the expressions $p_r = \dfrac{\partial T}{\partial \dot{q}_r} \quad (r = 1, 2, 3)$

are referred to as *generalized momenta.* One must be careful not to be misled by these names, as these generalized forces and momenta do not have the correct dimensions unless the corresponding coördinate has the dimensions of a length so that the corresponding A_r is a pure number.

The resolved part of the acceleration vector along the first coördinate line is

$$\frac{1}{\sqrt{A_1}}\left(\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_1} - \frac{\partial T_1}{\partial q_1}\right),$$

where

$$T_1 = \tfrac{1}{2}(A_1\dot{q}_1^2 + A_2\dot{q}_2^2 + A_3\dot{q}_3^2 + 2\,B_1\dot{q}_2\dot{q}_3 + 2\,B_2\dot{q}_3\dot{q}_1 + 2\,B_3\dot{q}_1\dot{q}_2)$$

is the kinetic energy of a particle of unit mass. The resolved part of the velocity vector in the same direction is

$$\frac{1}{\sqrt{A_1}}\frac{\partial T_1}{\partial \dot{q}_1}.$$

If we wish to analyze the acceleration vector into three component vectors of magnitudes (p, q, r), respectively, along the three coördinate lines, we find, on resolving along the first coördinate line, for example,

$$p + q\cos\gamma + r\cos\beta = \frac{1}{\sqrt{A_1}}\left(\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_1} - \frac{\partial T_1}{\partial q_1}\right),$$

where (α, β, γ) are the three angles made by the three coördinate directions through a point. We have, then, to determine (p, q, r), three equations of the type

$$\sqrt{A_1}p + \frac{B_3}{\sqrt{A_2}}q + \frac{B_2}{\sqrt{A_3}}r = \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_1} - \frac{\partial T_1}{\partial q_1}.$$

When the curvilinear coördinates are orthogonal, (B_1, B_2, B_3) are all zero, and the acceleration components coincide with the resolved parts of the acceleration vector along the coördinate directions. The simplicity thus introduced by the use of orthogonal coördinates is so great that they are used almost exclusively in the discussion of problems in mathematical physics.

EXAMPLES

1. *Plane polar coördinates* (ρ, ϕ). (See § 7, *a*.) Here $A_1 = 1$, $A_2 = \rho^2$, $B = 0$, $T_1 = \frac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2)$. The resolved parts of the velocity vector along the coördinate lines are, accordingly, $\dot{\rho}$ and $\rho\dot{\phi}$ respectively. The resolved parts of the acceleration vector are, respectively,

$$\ddot{\rho} - \rho\dot{\phi}^2 \quad \text{and} \quad \frac{d}{dt}(\rho^2\dot{\phi}).$$

2. *Cylindrical coördinates.* These differ from the preceding merely in having a third coördinate z, along which the resolved parts of the velocity and acceleration vectors are $\dot{z}$ and $\ddot{z}$ respectively.

3. *Space polar coördinates* (r, θ, ϕ). Here $A_1 = 1$, $A_2 = r^2$, $A_3 = r^2\sin^2\theta$, $B_1 = B_2 = B_3 = 0$, $T_1 = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$. The resolved parts of the velocity vector along the three coördinate directions are $(\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\phi})$ respectively, and the resolved parts of the acceleration vector are, respectively,

$$\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2,\quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\,\dot{\phi}^2,\right.$$
$$\left.\frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\,\dot{\phi})\right). \qquad \text{[See (30.8)]}$$

44. The cycloidal pendulum. We shall now deal with the problem of a particle of mass m moving on a smooth plane curve under the action of a constant external applied force $m\mathbf{g}$ of magnitude mg and in the plane of the curve. This constant force will usually be the weight of the particle, and we shall therefore call the constant direction of the force the downward vertical, the opposite direction being named the upward vertical. Denoting by ψ the angle made by the tangent to the curve with the horizontal, we have, on resolving $m\mathbf{A}$ and $\mathbf{F}$ along the tangent, the equation

$$m\ddot{s} = mg\cos\left(\psi + \frac{\pi}{2}\right) = -\,mg\sin\psi. \tag{44.1}$$

(See (5.6). The equation (44.1) is the special case of (43.4) which occurs when the parameter q is the arc distance s along the curve so that $A = 1$.)

Let us now seek the form of curve for which the motion is a simple harmonic motion whose equation is $m\ddot{s} = -\,n^2 s$. Comparing this with (44.1), we see that we must have $s = \frac{mg}{n^2}\sin\psi$. In order to obtain the parametric equations of this curve, we introduce a Cartesian reference frame whose x-axis is horizontal and whose y-axis is the upward vertical. Then

$$dx = ds\cos\psi = \frac{mg}{n^2}\cos^2\psi\,d\psi,$$

so that

$$x = \frac{mg}{4\,n^2}(2\,\psi + \sin 2\,\psi) + \text{constant},$$

and

$$dy = ds\sin\psi = \frac{mg}{n^2}\sin\psi\cos\psi\,d\psi,$$

or

$$y = \frac{-\,mg\cos 2\,\psi}{4\,n^2} + \text{constant}.$$

Choosing the origin on the curve at a point where $\psi = 0$, we find

$$\begin{aligned} x &= \frac{mg}{4\,n^2}(2\,\psi + \sin 2\,\psi), \\ y &= \frac{mg}{4\,n^2}(1 - \cos 2\,\psi). \end{aligned} \tag{44.2}$$

The curve is therefore a cycloid, which might be traced out by a point on the circumference of a circle of radius $\frac{mg}{4\,n^2} = a$, rolling on, and below, a horizontal line, so that the arches of the cycloid are concave upward. The frequency of the simple harmonic vibration is

$$\nu = \frac{n}{2\pi\sqrt{m}} = \frac{1}{4\pi}\sqrt{\frac{g}{a}}. \tag{44.3}$$

Since the evolute of a cycloid is a similar cycloid, this simple harmonic motion may be realized by suspending a small mass by means of a flexible but inextensible string from a fixed point at which two cycloidal "cheeks" meet. The fixed point is a cusp of the cycloid of which the two cheeks form a part, and the common tangent is vertical. As the string wraps and unwraps itself on the cycloidal cheeks the suspended mass moves in a cycloid (which is an involute of the cycloid of which the two cheeks form a part).

It will be observed that the frequency of the vibrations is independent of their amplitude. We have here, then, an apparatus which is *periodic,* that is, which furnishes us with a succession of equal intervals of time and therefore enables us to measure time, or, in other words, to give a number to any time interval. This apparatus is known as the cycloidal pendulum.

45. The simple pendulum. Here the particle moves in a circle under the action of a constant force mg in the plane of the circle. If the circle is a material one, we shall suppose it smooth, so that its reaction on the moving particle is always along the radius. If, on the other hand, the moving particle is attached to the fixed point by means of an inextensible string, the tension exerted by the string will also be along the radius, so that, in either case, the activity of the reaction (or tension) is everywhere zero. We may, therefore, in setting up the energy integral, consider merely the constant external applied force mg. Taking the z-axis of our reference frame parallel to the constant force and in the same direction, we see at once that the applied force is the negative gradient of the function $V = -mgz$. Thus we have the energy integral

$$\tfrac{1}{2}mv^2 - mgz = \text{constant},$$

or

$$v^2 = 2gz + \text{constant}.$$

Choosing the origin of our reference frame at the center of the circle, so that $z = l$ is the lowest point of the circle, and denoting the velocity at this lowest point by v_0, we have

$$v^2 = v_0^2 - 2\,g(l - z). \tag{45.1}$$

Introducing the angle θ made by the radius to the moving particle with the z-axis, we have $z = l\cos\theta$, $v^2 = l^2\dot{\theta}^2$; and our equation (45.1) becomes

$$\dot{\theta}^2 = \frac{v_0^2}{l^2} - 4\frac{g}{l}\sin^2\frac{\theta}{2}. \tag{45.2}$$

There are three distinct cases to be considered according as $v_0^2 \gtreqless 4\,gl$.

CASE I. $v_0^2 > 4\,gl$. Here $\dot{\theta}$ is never zero, and the particle moves steadily round the circle in the same direction. Assuming the motion to be counterclockwise, we have $\dot{\theta}$ positive and equal to $\frac{v_0}{l}\sqrt{\left(1 - k^2\sin^2\frac{\theta}{2}\right)}$, where $k^2 = \frac{4\,gl}{v_0^2} < 1$. Introducing the half-angle $\phi = \frac{\theta}{2}$, this gives

$$t = \frac{2\,l}{v_0}\int^{\phi}\frac{d\psi}{\sqrt{1 - k^2\sin^2\psi}} + \text{constant}.$$

Measuring t from the instant the particle is at the lowest point of the circle, we have

$$t = \frac{2\,l}{v_0}F(\phi, k),$$

where $F(\phi, k)$ is the integral

$$F(\phi, k) = \int_0^{\phi}\frac{d\psi}{\sqrt{1 - k^2\sin^2\psi}}. \tag{45.3}$$

This integral is known as Legendre's normal elliptic integral of the first kind; and its values may be found, for various values of ϕ and $k < 1$, in any table of elliptic integrals. Legendre's own tables give the values of $F(\phi, k)$ for every degree in ϕ and $\alpha = \text{arc sin}\, k$ to nine and ten decimal places (ten decimal places for α between 0° and 45°, and nine decimal places for α between 45° and 90°). The values of $K = F\left(\frac{\pi}{2}, k\right)$ are given in the same tables to twelve decimal places for every degree in α.

The time taken to reach the highest point of the circle is $\frac{2\,l}{v_0}K$, where $K = F\left(\frac{\pi}{2}, k\right)$ is the complete elliptic integral of the first kind. A good approximation to the value of K for a given value of k may be found, without recourse to the tables, as follows:

$$K=\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\psi)^{-\frac{1}{2}}d\psi$$

$$=\int_0^{\frac{\pi}{2}}\left(1+\frac{1}{2}k^2\sin^2\psi+\frac{1.3}{2.4}k^4\sin^4\psi+\cdots\right)d\psi$$

$$=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1.3}{2.4}\right)^2k^4+\left(\frac{1.3.5}{2.4.6}\right)^2k^6+\cdots\right]. \quad (45.4)$$

For reasonably small values of k the first few terms of this power series in k^2 suffice to give K to a high degree of approximation.

It follows at once, from the definition (45.3) of $F(\phi, k)$, that the time taken by the particle to go from the highest point of the circle to the lowest is the same as the time taken to ascend from the lowest point to the highest. We have, in fact,

$$\int_{\frac{\pi}{2}}^{\pi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}=-\int_{\frac{\pi}{2}}^{0}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}=\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}},$$

where $$\psi=\pi-\phi.$$

The time of a complete revolution is, accordingly, $\frac{4\,lK}{v_0}$. The same argument shows that if $\frac{\pi}{2}<\phi<\pi$,

$$F(\phi, k)=2\,K-F(\psi, k),$$

where $$\psi=\pi-\phi.$$

Hence, in determining the time taken to reach any point on the circle, it is unnecessary to use values of the argument ϕ which are greater than $\frac{\pi}{2}$. Moreover, since $\sin^2\phi$ has the period π, we have

$$F(\pi+\phi, k)=F(\phi, k)+F(\pi, k)=F(\phi, k)+2\,K.$$

Hence the motion is periodic, and the period is $T=\frac{4\,lK}{v_0}$.

CASE II. $v_0^2=4\,gl$. Here

$$t=\frac{2\,l}{v_0}\int_0^{\phi}\frac{d\psi}{\sqrt{1-\sin^2\psi}}=\frac{2\,l}{v_0}\int_0^{\phi}\sec\psi\,d\psi=\frac{2\,l}{v_0}\log(\sec\phi+\tan\phi).$$

The particle will reach any point short of the highest point $\phi = \frac{\pi}{2}$ of the circle, but it will never reach this highest point. The velocity is given by the formula (45.2) as

$$v_0 \sqrt{1 - \sin^2 \phi} = v_0 \cos \phi,$$

and this becomes infinitesimally small as the particle approaches the highest point. We may express ϕ as a function of t as follows:

From $$\sec \phi + \tan \phi = e^{\frac{v_0 t}{2l}}$$

we obtain, on taking the reciprocal of each side,

$$\sec \phi - \tan \phi = e^{-\frac{v_0 t}{2l}};$$

whence $$\sec \phi = \cosh \frac{v_0 t}{2l}. \qquad (45.5)$$

CASE III. $v_0^2 < 4gl$. Here the velocity of the particle vanishes before the particle reaches the highest point of the circle, and it will execute vibratory motions about the lowest point of the circle. Writing $\frac{v_0^2}{4gl} = k^2 < 1$, we have, from (45.2),

$$\dot{\theta}^2 = \frac{4g}{l}\left(k^2 - \sin^2 \frac{\theta}{2}\right);$$

so that $\dot{\theta}$ vanishes when $\theta = \pm \alpha$, where $\sin \frac{\alpha}{2} = k = \frac{v_0}{2}\sqrt{\frac{1}{gl}}$. Introducing the auxiliary angle ϕ defined by

$$\sin \phi = \frac{\sin \frac{\theta}{2}}{\sin \frac{\alpha}{2}},$$

we have $$\dot{\theta}^2 = \frac{4gk^2}{l}(1 - \sin^2 \phi),$$

or $$\dot{\theta} = \pm 2k\sqrt{\frac{g}{l}} \cos \phi.$$

When $\theta = \pm \alpha$ $\left(\text{that is, when } \phi = \pm \frac{\pi}{2}\right)$, $\dot{\theta}$ changes sign, passing through the value zero; and we may therefore remove the ambiguity in sign by agreeing that ϕ increases steadily. We have, then,

$$\dot{\theta} = 2k\sqrt{\frac{g}{l}} \cos \phi.$$

From the equation defining ϕ we derive $\cos\frac{\theta}{2}\dot{\theta} = 2k\cos\phi\,\dot{\phi}$. Upon substituting $\sqrt{1-k^2\sin^2\phi}$ for $\cos\frac{\theta}{2}$, we may eliminate θ and $\dot{\theta}$, obtaining the equation

$$\dot{\phi} = \sqrt{\frac{g}{l}}\sqrt{1-k^2\sin^2\phi};$$

whence

$$t = \sqrt{\frac{l}{g}}\int_0^{\phi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = \sqrt{\frac{l}{g}}\cdot F(\phi, k), \tag{45.6}$$

where we again agree to measure t from the lowest point of the circle, for which θ and ϕ are both zero.

The time from the lowest point reached by the particle to the highest point $\theta = \alpha$, or $\phi = \frac{\pi}{2}$, is $\sqrt{\frac{l}{g}}\cdot F\left(\frac{\pi}{2}, k\right) = \sqrt{\frac{l}{g}}\cdot K$. As θ diminishes again from α to 0, ϕ increases from $\frac{\pi}{2}$ to π, and the time of descent from the highest point $\theta = \alpha$ to the lowest point $\theta = 0$ is again $\sqrt{\frac{l}{g}}\cdot K$. As ϕ increases from π to 2π, θ takes negative values; and since $\sin^2\phi$ has the period π, the times taken to the highest point $\theta = -\alpha$, and from this back to the lowest point $\theta = 0$, are each equal to $\sqrt{\frac{l}{g}}\cdot K$. The motion is therefore periodic, and the period is

$$T = 4\sqrt{\frac{l}{g}}\cdot K. \tag{45.7}$$

This type of periodic motion differs from that of the cycloidal pendulum in that here the period depends on the amplitude α of the vibrations. If this amplitude is small, we may neglect, to a first approximation, the square and higher powers of $k = \sin\frac{\alpha}{2}$. We see then, from (45.4), that $K = \frac{\pi}{2}$, and the period of the *small* vibrations of a circular pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{45.8}$$

To get an idea of the closeness of the approximation of (45.8) to (45.7), let us consider the values 5°, 10°, and 30° of α. The corresponding values of $k = \sin\frac{\alpha}{2}$ are .0436, .0872, and .2589 respectively. The relative errors made by neglecting the squares and higher powers of k are therefore approximately .05, .19, and 1.68 per cent.

The nature of the motion has now been completely determined. If we wish to determine the reaction R of the supporting circle (or the tension of the supporting string), we resolve the equal vectors $m\ddot{\mathbf{r}}$ and $\mathbf{F}$ along the normal to the circle. The component of the acceleration along the normal being $\frac{v^2}{\rho}$ (see (5.6)), where ρ is the radius l of the circle, we have

$$\frac{mv^2}{l} = R - mg\cos\theta, \quad \text{or} \quad l\dot{\theta}^2 = \frac{R}{m} - g\cos\theta.$$

On using (45.2), we find

$$\frac{R}{m} = \frac{v_0{}^2}{l} + 3\,g\cos\theta - 2\,g. \tag{45.9}$$

If R cannot be negative, as is the case when the particle is merely supported by the circle or suspended by a string, the particle will leave the circle when R, as calculated by the formula (45.9), becomes negative on passing through the value zero. If the angle θ at which the particle leaves the circle be denoted by β, we have

$$\cos\beta = \frac{2 - \frac{v_0{}^2}{gl}}{3}. \tag{45.10}$$

If $4\,gl < v_0{}^2 < 5\,gl$, the particle will leave the circle before reaching the highest point of the circle, which point it would reach if R could become negative. If $v_0{}^2 < 4\,gl$, we have to compare the angle β given by (45.10) with the angle α which makes $\dot{\theta} = 0$, and which is given by $\cos\alpha = 1 - \frac{v_0{}^2}{2\,gl}$. We have $\cos\beta - \cos\alpha = \frac{v_0{}^2}{6\,gl} - \frac{1}{3}$. When $v_0{}^2 \leqq 2\,gl$, this is negative or zero, so that $\beta \geqq \alpha$, indicating that the particle *does not leave the circle.* The least value of R occurs when $\theta = \alpha$, and is given by the formula

$$\frac{R}{m} = g - \frac{v_0{}^2}{2\,l}.$$

When, on the other hand, $v_0{}^2 > 2\,gl$, the particle leaves the circle at the angle β given by (45.10), and it is seen at once that this angle is an obtuse one, its cosine being negative. The connection between the angle β for which $R = 0$ and the maximum angle α of the oscillatory motion is given by the simple formula

$$\cos\beta = \tfrac{2}{3}\cos\alpha.$$

46. The spherical pendulum. Here we have the problem of a particle constrained to move on a frictionless material sphere under a constant force mg; or the particle may be suspended from a fixed point O by means of an inextensible string, as in the case of the simple pendulum problem. The motion will no longer, in general, take place entirely in a vertical plane. Since both the constant applied force and the reaction of the sphere (or the tension of the supporting string) are coplanar with the vertical through the center of the sphere, we have an angular-momentum integral in addition to the energy integral. Taking the downward vertical through O as our z-axis, the resolved part along this axis of the angular-momentum vector $[\mathbf{r}.m\dot{\mathbf{r}}]$ is $m(x\dot{y} - y\dot{x})$, so that, on introducing the cylindrical coördinates (ρ, ϕ, z), we have

$$\rho^2\dot{\phi} = \text{constant}. \tag{46.1}$$

This might have been obtained from the general results of § 43 by using (ρ, ϕ) as the parameters q_1 and q_2 on the sphere. Then $T = \frac{m}{2}\left(\frac{l^2}{l^2 - \rho^2}\dot{\rho}^2 + \rho^2\dot{\phi}^2\right)\cdot$ Since both the applied force and the reaction of the sphere are perpendicular to the vector $\frac{\partial \mathbf{r}}{\partial \phi}$, $Q_2 = 0$, and the second Lagrangian equation (43.9) becomes

$$\frac{d}{dt}(m\rho^2\dot{\phi}) = 0, \quad \text{or} \quad \rho^2\dot{\phi} = \text{constant}.$$

The coördinate ϕ is an example of what is known as an ignorable coördinate, that is, a coördinate which does not occur explicitly in the expression for the kinetic energy T. The corresponding Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q$$

reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = Q;$$

so that when Q is a constant or, more generally, a function of t (the coördinate q not occurring in it), we have the integral

$$\frac{\partial T}{\partial \dot{q}} = \int^t Q\,dt + \text{constant}.$$

Since we are in possession of two integrals (namely, the energy integral and the angular-momentum integral) of the two

Lagrangian equations (43.8) and (43.9) for the two unknowns ρ and ϕ, it is unnecessary to consider these second-order differential equations further. We have merely to deal with the two first-order differential equations

$$v^2 = 2\,gz + h, \tag{46.2}$$

$$\rho^2\dot{\phi} = c, \tag{46.3}$$

where h is an energy constant and c is an angular-momentum constant. We use the relations $\rho^2 + z^2 = l^2$, $v^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2$ to eliminate ρ and ϕ, together with their derivatives, from the equations (46.2) and (46.3), thus obtaining a first-order differential equation for the variable z as a function of t. From $z^2 + \rho^2 = l^2$ we obtain $\rho\dot{\rho} + z\dot{z} = 0$, so that $\dot{\rho}^2 = \dfrac{z^2\dot{z}^2}{\rho^2} = \dfrac{z^2\dot{z}^2}{l^2 - z^2}$. Hence, on putting $\dot{\phi} = \dfrac{c}{\rho^2}$, we have $v^2 = \dfrac{l^2\dot{z}^2}{l^2 - z^2} + \dfrac{c^2}{l^2 - z^2}$. Then (46.2) gives

$$l^2\dot{z}^2 = (l^2 - z^2)(2\,gz + h) - c^2 = f_3(z), \tag{46.4}$$

where $f_3(z)$ denotes the cubic polynomial on the right-hand side.

It will be recalled that when dealing with the problem of simple harmonic motion in a straight line (see § 37), we had to integrate a similar equation,

$$m\dot{x}^2 = n^2(a^2 - x^2) = f_2(x).$$

This involved the evaluation of the integral $\displaystyle\int^x \frac{ds}{\sqrt{a^2 - s^2}}$, which we immediately wrote down as arc sin $\dfrac{x}{a}$ + constant. If we had not already been familiar with the properties of the inverse sine function, these properties could have been derived from a study of the integral $\displaystyle\int^x \frac{ds}{\sqrt{a^2 - s^2}}$. It appeared in (37.6) that it was simpler to regard the upper limit x as a function of the integral $\displaystyle\int^x \frac{ds}{\sqrt{a^2 - s^2}}$ than to regard the integral as a function of its upper limit. This change in the point of view is called the inversion of the integral. In planning a discussion of the integral $\displaystyle\int^x \frac{ds}{\sqrt{a^2 - s^2}}$ we should first make the substitution $x = a\xi$, $s = a\sigma$, and should then consider the equivalent integral $v = \displaystyle\int^\xi \frac{d\sigma}{\sqrt{1 - \sigma^2}}$, which we should call the standard form of the previous integral, since it is independent of the particular value assigned to the constant a. Accordingly, we should be interested in the behavior of the upper limit ξ as a function of the integral v. Writing $\xi = \sin v$, it would appear that ξ is a simply periodic function of v, the period being 2π. In this way the trigonometric functions would appear, and all their properties could be deduced without any reference to geometry. For example, it is apparent, from the fact that ξ satisfies the differential equation $\left(\frac{d\xi}{dv}\right)^2 = 1 - \xi^2$, that $\xi = \sin v$ cannot be greater than unity in absolute value, since the variables v and ξ are both real.

The procedure indicated here is exactly what must be done in dealing with the equation (46.4). The only difference is that the polynomial $f_3(z)$ of the *third* degree replaces the previous polynomial $f_2(z)$ of the *second* degree. The first thing to do is to reduce the equation to a standard form. Denoting the new dependent variable by p and the new independent variable by u, the standard form, due to Weierstrass, is

$$\left(\frac{dp}{du}\right)^2 = 4\,p^3 - g_2 p - g_3, \tag{46.5}$$

where g_2 and g_3 are combinations of the constants l, g, h, and c which occur in (46.4). It will be noticed that the second term is missing from the cubic in (46.5) and that the coefficient of the cube is 4. In order to remove the square from the cubic in (46.4), we have to write $z = \alpha p - \frac{h}{6\,g}$, where α is an arbitrary constant, and we also write $t = u + \beta$, where β is an undetermined constant. We find, on substituting these values in (46.4), that

$$l^2\alpha^2\left(\frac{dp}{du}\right)^2 = -\,2\,g\alpha^3p^3 + \cdots.$$

On comparing this with (46.5), we have $\alpha = \frac{-2\,l^2}{g}\cdot$ We find, then, that the substitutions,

$$\begin{aligned} z &= -\frac{2\,l^2}{g}p - \frac{h}{6\,g}, \\ t &= u + \beta, \quad (\beta \text{ an undetermined constant}) \end{aligned} \tag{46.6}$$

reduce the equation (46.4) to the standard form (46.5).

Before proceeding to a discussion of the function $p(u)$ defined by (46.5), an important property of the motion may be deduced at once from the equation (46.4). First of all we observe that the cubic polynomial $f_3(z)$ has three real zeros, no matter what numeric values may be assigned to the constants l, g, h, and c. For if c be zero, $f_3(z)$ has the zeros $\pm l$, $\frac{-h}{2g}$; whereas if $c \neq 0$, $f_3(-\infty)$ is positive and $f_3(\pm l)$ is negative. Hence $f_3(z)$ has a zero between $-l$ and $-\infty$. Moreover, if z_0 is the initial value of z, $f_3(z_0)$, which is identically equal to $l^2\dot{z}_0^2$, is either positive or zero, so that $f_3(z)$ has a zero in each of the intervals $-l < z \leqq z_0$, and $z_0 \leqq z < l$. Denoting the three real zeros of $f_3(z)$, when arranged in ascending order of magnitude, by z_1, z_2, and z_3 respectively, we have

$$-\infty < z_1 < -l < z_2 \leqq z_0 \leqq z_3 < l.$$

We may rewrite (46.4) in the form

$$l^2\dot{z}^2 = 2g(z - z_1)(z - z_2)(z_3 - z).$$

Since this product cannot be negative, and since $z - z_1$ is necessarily positive (z being constrained, by the nature of the problem, to lie in the interval $-l \leqq z \leqq l$), the remaining factors $z - z_2$ and $z_3 - z$ cannot have opposite signs. Hence z must lie in the interval $z_2 \leqq z \leqq z_3$, and we see that *the entire motion of the particle takes place between the two horizontal circles* $z = z_2$ *and* $z = z_3$ *on the sphere.*

a. The conical pendulum. We are now prepared to answer the question: What must be the relation between the constants h and c of integration if the particle is to move in a horizontal circle? The two zeros z_2 and z_3 of $f_3(z)$ must coincide; conversely, we know that if $f_3(z)$ has two coincident zeros, they must be z_2 and z_3, since $z_1 < -l$ and $-l < z_2 < z_3 < l$. The condition that $f_3(z)$ should have a double zero $z = z_2$ may be found by eliminating z_2 between

$$f_3(z_2) = (2gz_2 + h)(l^2 - z_2^2) - c^2 = 0$$

and its derivative

$$f'_3(z_2) = 2g(l^2 - z_2^2) - 2z_2(2gz_2 + h) = 0.$$

For a given value of z_2, we find $h = \frac{g}{z_2}(l^2 - 3z_2^2)$. Upon substituting this in $f_3(z_2) = 0$, we find

$$c^2 = \frac{g}{z_2}(l^2 - z_2^2)^2. \qquad (46.7)$$

This equation shows that z_2 must be positive, so that *the particle will move in a horizontal circle only when this horizontal circle is below the center of the sphere.*

In order to determine the initial horizontal velocity which will make the particle describe a horizontal circle, we substitute the value of h, just found, in (46.2); and we get

$$v^2 = \frac{g(l^2 - z_2^2)}{z_2},$$

or, in terms of the angle θ,

$$v = \sin\theta\sqrt{\frac{gl}{\cos\theta}}. \tag{46.8}$$

Since the velocity is constant, the motion is periodic, the period being $T = \frac{2\pi l \sin\theta}{v} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$. This motion of the particle in a horizontal circle is called the *conical pendulum motion,* and for small values of θ the period is sensibly the same as that of the small oscillations of a simple pendulum of length l.

Returning now to a consideration of (46.5), let us denote by e_1, e_2, and e_3 the zeros of $4p^3 - g_2p - g_3$ which correspond to the zeros z_1, z_2, and z_3, respectively, of $f_3(z)$. We have, then,

$$z_1 = -\frac{2l^2}{g}e_1 - \frac{h}{6g}, \quad z_2 = -\frac{2l^2}{g}e_2 - \frac{h}{6g}, \quad z_3 = -\frac{2l^2}{g}e_3 - \frac{h}{6g}.$$

As z_1, z_2, z_3 are arranged in ascending order of magnitude, e_1, e_2, e_3 are arranged in descending order of magnitude and are such that their sum is zero. As we have already considered the conical pendulum, for which $z_2 = z_3$, we may suppose $z_1 < z_2 < z_3$; whence $e_1 > e_2 > e_3$. The function $p(u)$ defined by the differential equation

$$\left(\frac{dp}{du}\right)^2 = 4(p - e_1)(p - e_2)(p - e_3) \tag{46.5a}$$

has several important and characteristic properties which are brought clearly to light only when we allow the independent variable u to assume complex values in addition to real values. The most important of these properties is one which recalls the analogy between the p-function and the trigonometric sine function; namely, the property that $p(u)$ is periodic in u but has two distinct periods, one of which is real and the other a pure imaginary. These two periods are denoted by $2\,\omega_1$ and $2\,\omega_3$ respectively, so that

$$p(u + 2\,\omega_1) = p(u) = p(u + 2\,\omega_3)$$

or, more generally, $p(u + 2\,m\omega_1 + 2\,n\omega_3) = p(u)$, where m and n are any integers. It is convenient to construct the period rectangle whose corners are at the points ($u=0, u=2\,\omega_1, u=2\,\omega_1+2\,\omega_3$, $u = 2\,\omega_3$) in the plane of the complex variable u. The dotted sides of the rectangle not being supposed to belong to it, the function $p(u)$ takes every value, real or complex, exactly twice in the rectangle. Moreover, $p(u)$ has, when developed as a power series in u, the expansion

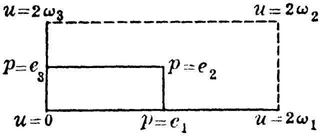


FIG. 26

$$p(u) = \frac{1}{u^2} + c_2u^2 + c_3u^4 \cdots, \tag{46.9}$$

where c_2, c_3, etc. are certain functions of g_2 and g_3. Hence $p(u)$ is an even function of u. Denoting, then, the derived period $2\,\omega_1 + 2\,\omega_3$ by $2\,\omega_2$, we have

$$\begin{aligned} p(\omega_2 + u) &= p(-\,\omega_2 + u) && \text{[Since } 2\,\omega_2 \text{ is a period]} \\ &= p(\omega_2 - u). && \text{[Since } p(u) \text{ is an even function]} \end{aligned}$$

This shows that the two points in the period rectangle at which $p(u)$ takes equal values are symmetrically situated with respect to the middle point $u = \omega_2$ of the rectangle. Again $p(u)$ takes real values along the sides and middle lines of the period rectangle; and since it takes each real value twice along these lines, it cannot be real at any point of the period rectangle which does not lie on these lines. If we consider the quarter-period rectangle $(0, \omega_1, \omega_2, \omega_3)$, $p(u)$ takes every real value just once on its boundary. As u goes from zero to ω_1, $p(u)$ decreases from $+\infty$ to e_1. As u goes on from ω_1 to ω_2, then to ω_3, and back to zero, $p(u)$ decreases steadily from e_1 to e_2, then to e_3, and finally to $-\infty$. We have already seen that, in our problem of the spherical pendulum, $p(u)$ is constrained to lie in the interval $e_3 \leqq e_2$. Hence u must lie on that middle line of the period rectangle which is the join of ω_3 and $2\,\omega_1 + \omega_3$. Since $u = t + \beta$, where t is necessarily real and β is an undetermined constant, we see that β must be the pure imaginary half-period ω_3. This gives the final result

$$z = -\frac{h}{6\,g} - \frac{2\,l^2}{g}\,p(t + \omega_3) \tag{46.10}$$

for the depth of the moving particle below the center of the sphere. This expression, while theoretically complete, is not in a satisfactory form for the purposes of calculation. It follows, from (46.9), that the value of $p(u)$ depends on three arguments, u, g_2, and g_3; so that if one were to construct a table of its values, one would require a *triple*-entry table. This difficulty may, however, be avoided by inverting (46.10) and expressing t by means of Legendre's normal elliptic integral of the first kind, $F(\phi, k)$ (see (45.3)). This is best done with the aid of the Jacobian elementary elliptic functions sn, cn, and dn, which may be defined in terms of $p(u)$ as follows. Writing $v = \sqrt{e_1 - e_3}.t$, we put

$$\operatorname{sn} v = \sqrt{\frac{e_1 - e_3}{p(t) - e_3}}, \quad v = \sqrt{e_1 - e_3}.t. \tag{46.11}$$

As t varies from 0 to ω_1, $\operatorname{sn} v$ varies from 0 to 1; and as t varies from ω_1 to $2\,\omega_1$, $\operatorname{sn} v$ retraces its values from 1 to 0. Then, as t continues to increase from $2\,\omega_1$ to $3\,\omega_1$, $\operatorname{sn} v$ varies from 0 to -1; and, finally, as t increases from $3\,\omega_1$ to $4\,\omega_1$, $\operatorname{sn} v$ retraces its values from -1 to 0. As t keeps on increasing, $\operatorname{sn} v$ again goes through the cycle 0, 1, 0, -1, so that it is a periodic function with the period $4\,\omega_1\sqrt{e_1 - e_3}$. Since $\operatorname{sn} v$ is not greater than unity in absolute value, we may put

$$\operatorname{sn} v = \sin \phi, \tag{46.12}$$

where ϕ is real. Upon writing $\cos\phi = \operatorname{cn} v$, we have

$$\operatorname{cn} v = \sqrt{\frac{p(t) - e_1}{p(t) - e_3}}.$$

Denoting the similar expression $\sqrt{\dfrac{p(t) - e_2}{p(t) - e_3}}$ by $\operatorname{dn} v$, it readily follows that $\operatorname{dn}^2 v = 1 - k^2 \operatorname{sn}^2 v$, where

$$k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}} < 1,$$

and is known as the modulus of the elliptic functions.

If we differentiate (46.12), we find $\frac{d}{dv}\operatorname{sn} v = \cos\phi\,\frac{d\phi}{dv}$; and, using (46.11), this is seen to be equivalent to

$$\frac{-1}{2[p(t)-e_3]^{\frac{3}{2}}}\,\frac{dp}{dt} = \cos\phi\,\frac{d\phi}{dv}.$$

But $\frac{dp}{dt} = \pm 2\sqrt{[p(t)-e_1][p(t)-e_2][p(t)-e_3]}$; [By (46.5 *a*)]

and a consideration of (46.9) shows that we must use the negative sign. We have, then,

$$\operatorname{cn} v \operatorname{dn} v = \cos\phi\,\frac{d\phi}{dv} = \frac{d}{dv}\operatorname{sn} v,$$

or
$$dv = \frac{d\phi}{\operatorname{dn} v} = \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

Hence
$$v = \int_0^{\phi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = F(\phi, k), \tag{46.13}$$

where the lower limit is determined by the fact that when $v = 0$, $t = 0$, and $\operatorname{sn} v = 0$, so that $\sin\phi = 0$, and we may take $\phi = 0$. We see, then, that ϕ is the amplitude of the elliptic integral $F(\phi, k)$ which gives v, and it is for this reason known as the amplitude of the elliptic functions.

By using the elliptic-function formula

$$[p(t+\omega_3)-e_3][p(t)-e_3] = (e_1-e_3)(e_2-e_3),$$

we may replace the p-function which occurs in (46.11) by the depth z which is connected with the p-function by the equation (46.10). Thus

$$\operatorname{sn} v = \sqrt{\frac{e_1-e_3}{p(t)-e_3}} = \sqrt{\frac{p(t+\omega_3)-e_3}{e_2-e_3}} = \sqrt{\frac{z_3-z}{z_3-z_2}}.$$

The depth z of the particle on the sphere is, then, connected with the time by the formulas

$$\phi = \arc\sin\sqrt{\frac{z_3-z}{z_3-z_2}},$$

$$t = \frac{v}{\sqrt{e_1-e_3}} = \frac{F(\phi, k)}{\sqrt{\frac{g}{2l^2}(z_3-z_1)}}, \tag{46.14}$$

$$k = \sqrt{\frac{e_2-e_3}{e_1-e_3}} = \sqrt{\frac{z_3-z_2}{z_3-z_1}}.$$

The time is counted from the position of the particle for which $\phi = 0$, that is, from the lowest point $z = z_3$ reached by the particle. The time taken to reach the higher circle $z = z_2$ is given, on putting $\phi = \frac{\pi}{2}$, as

$$\frac{T}{2} = \frac{K}{\sqrt{\frac{g}{2\,l^2}(z_3 - z_1)}},$$

and the time taken to fall back to the lower circle is again equal to this. The vertical motion of the particle on the sphere is periodic, the period being

$$T = \frac{2\sqrt{2}\,lK}{\sqrt{g(z_3 - z_1)}}. \tag{46.15}$$

47. The small oscillations of a spherical pendulum about a conical pendulum motion. In the conical pendulum motion we have $z_2 = z_3 = l\cos\theta_0$, say, and as $z_1 + z_2 + z_3 = \frac{-h}{2\,g}$ in general, we find $z_1 = -\frac{h}{2\,g} - 2\,l\cos\theta_0$. Moreover, $h = \frac{g}{z_2}(l^2 - 3\,z_2{}^2)$ in this case, so that

$$z_1 = -\frac{l}{2}\left(\cos\theta_0 + \frac{1}{\cos\theta_0}\right) \quad \text{and} \quad z_3 - z_1 = \frac{l(1 + 3\cos^2\theta_0)}{2\cos\theta_0}.$$

If, now, we suppose the particle to be slightly disturbed from its steady conical pendulum motion, so that $z_3 - z_2$ is infinitesimal, we may neglect all but the first term in the series (45.4) and put $K = \frac{\pi}{2}$. We find that the particle executes a simple periodic vertical vibration about its previous steady conical motion, the period being given by (46.15) as

$$T = \frac{\pi}{\sqrt{\frac{lg(1 + 3\cos^2\theta_0)}{4\,l^2\cos\theta_0}}} = 2\,\pi\sqrt{\frac{l}{g}\cdot\frac{\cos\theta_0}{1 + 3\cos^2\theta_0}}. \tag{47.1}$$

This *nutation* involves a periodic vibration about the constant angular velocity $\dot{\phi}$ of the steady conical pendulum motion. For the angular-momentum integral $\rho^2\dot{\phi} = c$, on expanding $\dot{\phi}$ as a power series in $(\theta - \theta_0)$, gives

$$\dot{\phi} = \frac{c}{l^2\sin^2\theta} = \frac{c}{l^2\sin^2\theta_0}[1 - 2\cot\theta_0(\theta - \theta_0) + \cdots].$$

Writing $\theta - \theta_0 = A \sin 2\pi\nu t$, where

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{l} \cdot \frac{1 + 3\cos^2\theta_0}{\cos\theta_0}}$$

is the frequency of the nutation, we have

$$\begin{aligned}\phi &= \frac{c}{l^2 \sin^2\theta_0}\left[t + \frac{2A\cot\theta_0}{2\pi\nu}(\cos 2\pi\nu t - 1)\right], \text{ approximately}\\ &= \frac{ct}{l^2\sin^2\theta_0} + \frac{2A\cot\theta_0}{\sqrt{1 + 3\cos^2\theta_0}}(\cos 2\pi\nu t - 1). \qquad (47.2)\end{aligned}$$

We measure ϕ from $t = 0$ and avail ourselves of (46.7) to give $c = l^2 \sin^2\theta_0\sqrt{\frac{g}{l\cos\theta_0}}$ for the conical pendulum motion.

The formula (47.2) gives the *precessional* motion of a spherical pendulum which is executing small vibrations about a steady conical pendulum motion. The discussion of the precessional motion of a spherical pendulum in general is somewhat complicated, as it involves elliptic integrals of the third kind, that is, of the type $\int_0^\phi \frac{d\psi}{(1 + n\sin^2\psi)\sqrt{1 - k^2\sin^2\psi}}$, where n is a constant. For the details of the discussion the reader who is interested is referred to any treatise on elliptic functions.

48. Determination of the reaction of the sphere. Since the direction cosines of the inward normal to the sphere at any point (x, y, z) are $\left(-\frac{x}{l}, -\frac{y}{l}, -\frac{z}{l}\right)$, we have the three equations

$$m\ddot{x} = \frac{-Rx}{l}, \quad m\ddot{y} = \frac{-Ry}{l}, \quad m\ddot{z} = mg - \frac{Rz}{l}.$$

In order to eliminate the acceleration components, we differentiate the equation $x^2 + y^2 + z^2 = l^2$ twice with respect to the time. Thus we get

$$x\dot{x} + y\dot{y} + z\dot{z} = 0,$$

and

$$x\ddot{x} + y\ddot{y} + z\ddot{z} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 0,$$

or

$$x\ddot{x} + y\ddot{y} + z\ddot{z} = -v^2.$$

The elimination of $(\ddot{x}, \ddot{y}, \ddot{z})$ gives

$$R = m\frac{v^2 + gz}{l}. \qquad (48.1)$$

This reduces, by use of the energy integral (46.2), to

$$R = m\frac{3gz + h}{l}. \qquad (48.1\,a)$$

The particle will leave the supporting sphere at the height $z = \frac{-h}{3g}$ if it reaches this height, that is, if $\frac{-h}{3g}$ is in the interval $z_2 < \frac{-h}{3g} < z_3$. This will happen if $h < 3gl$ and if $f_3\left(\frac{-h}{3gl}\right)$ is positive.

EXAMPLES

1. The two bounding circles for the spherical pendulum are at depths $z_2 = \frac{3l}{5}$, $z_3 = \frac{4l}{5}$. Find the depth at any time t after the particle passes through a point of lowest depth, and determine the reaction R of the sphere.

Solution. We first find the value of the constant h which occurs in the energy integral $v^2 = 2gz + h$. At each of the bounding circles $\dot{z} = 0$, so that $v^2 = \rho^2\dot{\phi}^2 = (l^2 - z^2)\dot{\phi}^2$, and $\rho^2\dot{\phi} = c$, so that, at the circle z_2, $v_2 = \rho_2\dot{\phi}_2 = \frac{c}{\rho_2} = \frac{5c}{4l}$. Similarly, at the lower bounding circle, $v_3 = \frac{5c}{3l}$; and we have the two equations

$$\frac{25c^2}{16l^2} = \frac{6gl}{5} + h, \quad \frac{25c^2}{9l^2} = \frac{8gl}{5} + h.$$

Eliminating c, we have $h = -\frac{24gl}{35}$. Hence $z_1 + z_2 + z_3$, being equal to $-\frac{h}{2g}$, has the value $\frac{12l}{35}$; whence $z_1 = -\frac{37l}{35}$. From this we find at once $k = \sqrt{\frac{z_3 - z_2}{z_3 - z_1}} = \sqrt{\frac{7}{65}}$; and (46.14) gives

$$\frac{z_3 - z}{z_3 - z_2} = \sin^2\phi = \operatorname{sn}^2 v = \operatorname{sn}^2\sqrt{\frac{g}{2l^2}(z_3 - z_1)} \cdot t.$$

Upon substituting the values of (z_1, z_2, z_3) obtained above, we find

$$z = \frac{l}{5}\left(4 - \operatorname{sn}^2\sqrt{\frac{13g}{14l}} \cdot t\right); \quad \left(\text{mod.}\sqrt{\frac{7}{65}}\right)$$

Moreover, $$\frac{R}{m} = \frac{3gz + h}{l} = 3g\left(\frac{4}{7} - \frac{l}{5}\operatorname{sn}^2\sqrt{\frac{13g}{14l}} \cdot t\right).$$

2. The particle being projected initially in a horizontal direction, determine whether it will rise or fall; and if it rises, determine the greatest height to which it will rise.

Solution. Denote the initial depth below the center by z_0, and the initial horizontal velocity by V so that $V^2 = 2gz_0 + h$. Now z_0 is a zero of the cubic polynomial $f_3(z) = (2gz + h)(l^2 - z^2) - c^2$ of (46.4), so that the other two zeros are found, on dividing $f_3(z)$ by $z - z_0$ and using $V^2 = 2gz_0 + h$, to satisfy the quadratic equation

$$2gz^2 + V^2z + z_0V^2 - 2gl^2 = 0.$$

The greater of the two roots of this quadratic gives the second bounding circle, and the particle will rise or fall according as z_0 is greater or less than this root $\frac{(-V^2+\sqrt{V^4-8\,gz_0V^2+16\,g^2l^2})}{4\,g}$. Thus the particle rises or falls according as

$$z_0V^2 \gtrless g(l^2-z_0^2).$$

The particle will always fall if z_0 is zero or negative, that is, if the initial position of the particle is not below the center of the sphere. If z_0 is positive, the inequality $V^2 > \frac{g(l^2-z_0^2)}{z_0}$ determines the velocities necessary to make the particle rise. If the particle rises and does not leave the sphere, the greatest height reached is $z_2 = \frac{-V^2+\sqrt{V^4-8\,gz_0V^2+16\,g^2l^2}}{4\,g}$. If the particle leaves the sphere before reaching the upper bounding circle, the greatest height reached is $z = -\frac{h}{3\,g} = \frac{2\,z_0}{3} - \frac{V^2}{3\,g}$.

BIBLIOGRAPHY

Any of the texts mentioned on page 103 may be referred to.

A particularly good account of the spherical pendulum, with instructive illustrations, may be found in

Webster, A. G. Dynamics of a Particle. Leipzig, 1912.

We recommend the following texts on the theory of elliptic functions:

Appell, P., et Lacour, E. Fonctions elliptiques et applications. Paris, 1897.

Whittaker, E. T., and Watson, G. N. A Course of Modern Analysis. Cambridge (England), 1915.

This contains a valuable summary of the theory. Indeed, if we had to recommend a single book on mathematics for a student of theoretical physics, we should choose this text.

EXERCISES

1. A particle is projected horizontally along the interior surface of a fixed smooth hemisphere, the axis of which is vertical and whose vertex is downward. Given the point of projection, determine the velocity required in order that the particle may ascend exactly to the rim of the hemisphere.

2. Prove that a seconds pendulum, when taken to the top of a mountain h miles high, will lose nearly 21.6 h beats in a day.

3. The point of suspension of a simple pendulum has a horizontal motion expressed by $x = a \cos mt$. Find the effect upon the motion of the pendulum when $m^2 = \frac{g}{l}$, approximately.

4. A train is running smoothly along a curve at the rate of 60 miles an hour, and a pendulum which would ordinarily oscillate seconds is observed to oscillate 121 times in 2 minutes. Show that the radius of the curve in which the train is running is very nearly $\frac{1}{4}$ mile.

5. The bob of a pendulum which is hung close to the face of a cliff is attracted to the cliff with a horizontal force of magnitude f per unit mass. Show that the time of a beat is $\frac{\pi l^{\frac{1}{2}}}{(g^2 + f^2)^{\frac{1}{4}}}$, where l is the length of the pendulum.

6. Find the greatest angle through which a person can oscillate in a swing whose ropes can support a tension equal to twice the person's weight.

7. Prove that if a seconds pendulum makes a complete finite oscillation in 4 seconds the angle α of a quarter-swing is about 160°.

8. A pendulum oscillates in a medium of which the resistance per unit of mass is kv^2. Prove that when powers of the arc above the first are neglected, the period is the same as in the absence of resistance, but that the time of descent exceeds that of ascent by $\frac{2}{3} k\alpha \left(\frac{l^3}{g}\right)^{\frac{1}{2}}$, where α is the angular amplitude of the descent and l is the length of the pendulum.

9. A particle moves on a smooth cycloid, whose axis is vertical and whose vertex is downward, under gravity and under a resistance varying as the velocity. Prove that the time of falling from any point to the vertex is independent of the starting point.

10. A particle is projected horizontally from the lowest point of a smooth elliptic arc, whose major axis $2a$ is vertical, and moves under gravity along the concave side. Prove that it will leave the curve if the velocity of projection lies between $(2ga)^{\frac{1}{2}}$ and $[ga(5 - e^2)]^{\frac{1}{2}}$.

11. A smooth horizontal circular wire rotates uniformly about a point in its plane. Prove that the motion of a bead in the wire will be the same as that of the bob of a simple pendulum.

12. One end of an elastic string, natural length a, modulus λ, is tied to a fixed point on a smooth horizontal table, and the other end is tied to a particle of mass m which rests on the table. If the mass is pulled to a distance $2a$ from the other end of the string and is then let go, show that it will return to its original position at regular intervals $2(\pi + 2)\left(\frac{am}{\lambda}\right)^{\frac{1}{2}}$.

13. A particle begins to move from a distance a toward a fixed center which repels according to the law μr. If its initial velocity is $(\mu a)^{\frac{1}{2}}$, show that it will continually approach the fixed center but will never reach it.

14. A particle of mass m moves on the inner surface of a cone of revolution, whose semivertical angle is α, under the action of a repulsive force $\frac{m\mu}{r^3}$ from the axis. The angular momentum of the particle about the axis is $m\mu^{\frac{1}{2}} \tan \alpha$. Prove that its path is an arc of a hyperbola whose eccentricity is $\sec \alpha$.

15. Discuss the motion of a heavy particle on a smooth paraboloid of revolution placed with its axis vertical and its vertex downward. Show that the pressure at any point is inversely proportional to the radius of curvature of the parabola.

CHAPTER VI

CENTRAL FORCES; NEWTON'S LAW OF UNIVERSAL GRAVITATION

49. Motion under a central force. We have already seen in § 37 that the path traced out by a particle moving under the action of a central force is a plane curve. Introducing polar coördinates (r, θ) in the plane of this curve, the origin being at the center O, we have, on denoting the magnitude of the central force by $mf(r)$, the following two equations, expressing the accelerations along and perpendicular to $\mathbf{r}$ (see (30.9)),

$$\ddot{r} - r\dot{\theta}^2 = -f(r), \tag{49.1}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{49.2}$$

The magnitude of the central force is assumed to be independent of θ. The central force is taken as acting toward the center O. If it acts away from O, we have merely to change the sign before $f(r)$.

If A denotes the area swept out by the radius vector to the moving particle, measured from any convenient initial radius vector, the equation (49.2) is equivalent to

$$\frac{d}{dt}\left(2\frac{dA}{dt}\right) = 0, \quad \text{or} \quad \frac{dA}{dt} = \text{constant}.$$

We have, then, the result that in motion under a central force the radius vector sweeps out areas at a constant rate. This result is a special case of the angular-momentum-integral theorem stated at the end of § 42. For $r^2\dot{\theta} = 2\dfrac{dA}{dt}$ is the magnitude of the angular momentum of a particle of unit mass about the point O through which the force vector always acts, and is therefore a constant.

We may use the angular-momentum integral $r^2\dot{\theta} = C$ to remove the time variable from (49.1), thus obtaining a differential equation connecting r and θ, that is, the differential equation of the path. Thus

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{C}{r^2}\frac{dr}{d\theta}, \quad \text{and} \quad \ddot{r} = \frac{C}{r^2}\frac{d}{d\theta}\left(\frac{C}{r^2}\frac{dr}{d\theta}\right) = -C^2u^2\frac{d^2u}{d\theta^2},$$

where $u = \frac{1}{r}\cdot$ Moreover, $r\dot{\theta}^2 = \frac{C^2}{r^3} = C^2u^3$, so that (49.1) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{f}{C^2u^2}, \tag{49.3}$$

where f is the magnitude, per unit of mass, of the applied force toward the center.

The simplest case, and also that of greatest importance, is that where the right-hand side of (49.3) is constant. We have, then, $f = ku^2$, so that the applied central force is inversely proportional to the square of the distance from the fixed center O. Our equation (49.3) becomes

$$\frac{d^2}{d\theta^2}\left(u - \frac{k}{C^2}\right) + \left(u - \frac{k}{C^2}\right) = 0,$$

of which the general solution is

$$u - \frac{k}{C^2} = A\cos(\theta - \delta), \tag{49.4}$$

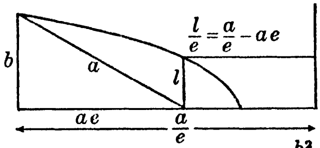
where A and δ are constants of integration. This is the polar equation of a conic of which the origin O is a focus and for which the line through O, $\theta = \delta$, is an axis of symmetry. In fact, the polar equation of a conic, with the origin at a focus and with the polar axis an axis of symmetry, is $r = \frac{l}{1 + e\cos\theta}$, where l is the semichord through the focus perpendicular to the axis, and e is the eccentricity. Comparing this with (49.4), we have $l = \frac{C^2}{k}$, $e = \frac{AC^2}{k}$; so that we may rewrite (49.4) in the form

$$u = \frac{k}{C^2}(1 + e\cos\theta), \tag{49.4 a}$$

where the axis of the conic through the focus has been taken as the polar axis $\theta = 0$.

If the conic traced out by the particle happens to be an ellipse, the time T taken to make a complete circuit is readily found. Since $2\frac{dA}{dt} = C$, we have $A = \frac{1}{2}Ct$, where A is the area swept out by the radius vector measured from the radius vector corresponding to $t = 0$. If a is the semimajor axis of the ellipse, the area of the ellipse $= \pi a^2 \sqrt{1 - e^2}$, and $l = a(1 - e^2)$; so that

$$T = \frac{2\,\pi a^2 \sqrt{1 - e^2}}{C} = \frac{2\,\pi a^{\frac{3}{2}} l^{\frac{1}{2}}}{C} = \frac{2\,\pi a^{\frac{3}{2}}}{k^{\frac{1}{2}}}. \tag{49.5}$$



$b^2 = a^2(1 - e^2) \quad l = a(1 - e^2) = \frac{b^2}{a}$

FIG. 27

In deriving this formula the accompanying figure may prove useful. The question as to whether the conic is an ellipse, parabola, or hyperbola — that is, whether e is less than, equal to, or greater than unity — may be answered by considering the energy integral. Thus we have, on multiplying (49.1) by $\dot{r}$ and integrating with respect to t, after elimination of $\dot{\theta}$ by means of the angular-momentum integral $r^2\dot{\theta} = C$, the energy integral

$$\dot{r}^2 + \frac{C^2}{r^2} = -2\int^r f(s)ds + \text{constant}. \tag{49.6}$$

Writing $r = \frac{1}{u}$ and $f(r) = \frac{k}{r^2} = ku^2$, this becomes

$$\frac{\dot{u}^2}{u^4} + C^2u^2 = 2\,ku + \text{constant}.$$

The expression on the left-hand side of this equation is the square of the velocity of the particle, and so the constant on the right-hand side is the square of the velocity when $u = 0$, that is, when $r = \infty$. Denoting it, therefore, by $v_\infty{}^2$, we have

$$\frac{\dot{u}^2}{u^4} + C^2u^2 = 2\,ku + v_\infty{}^2. \tag{49.7}$$

It readily follows, from (49.4 a) and (49.7), that

$$v_\infty{}^2 = \frac{k^2(e^2 - 1)}{C^2}. \tag{49.8}$$

Hence the conic is an ellipse, parabola, or hyperbola according as $v_\infty{}^2$ is negative, zero, or positive, that is, according as the square of the velocity at any point is less than, equal to, or greater than $\frac{2\,k}{r}$. If k is negative, so that the central force is a repulsive one, the path is always a hyperbola. The meaning of $v_\infty^2 < 0$ is that in this case the conic is an ellipse and the particle never goes off to ∞. The expression v_∞^2 is simply an abbreviation for $v^2 - \frac{2k}{r}$. The potential energy, relative to ∞, is $\frac{-mk}{r}$.

EXAMPLE

What is the form of the path when there is superimposed on the inverse-square central force a central force directed toward the same center and inversely proportional to the cube of the distance?

Solution. Denoting the magnitude of the additional force by $m\epsilon u^3$, our equation (49.3) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{C^2} + \frac{\epsilon u}{C^2},$$

or

$$\frac{d^2}{d\theta^2}\left(u - \frac{k}{C^2 - \epsilon}\right) + \left(1 - \frac{\epsilon}{C^2}\right)\left(u - \frac{k}{C^2 - \epsilon}\right) = 0,$$

and its general integral is

$$(1)\quad u = \frac{k}{C^2 - \epsilon}\left[1 + e\cos\sqrt{1 - \frac{\epsilon}{C^2}}\cdot(\theta - \delta)\right] \quad \text{when} \quad \epsilon < C^2,$$

$$(2)\quad u = A + B\theta + \frac{k}{2C^2}\theta^2 \quad \text{when} \quad \epsilon = C^2,$$

$$(3)\quad u = \frac{k}{\epsilon - C^2}\left[e'\cosh\sqrt{\frac{\epsilon}{C^2} - 1}\cdot(\theta - \delta)\right] \quad \text{when} \quad \epsilon > C^2.$$

These are, in general, not closed curves. In order that u and $\frac{du}{d\theta}$ may repeat their values, we must limit ourselves to case (1), where $\epsilon < C^2$; θ must then increase by $\frac{2\pi}{\sqrt{1 - \frac{\epsilon}{C^2}}}$ or $2\pi\left[1 + \left(\frac{\epsilon}{2C^2}\right) + \frac{1.3}{2.4}\left(\frac{\epsilon}{C^2}\right)^2 + \cdots\right]$. If $\frac{\epsilon}{C^2}$ is small, and if the curve which would be described under the action of the inverse-square force alone is an ellipse, we may regard the curve (1) as closely approximated by a slowly revolving ellipse. During each revolution of the particle in its elliptical orbit the major axis of the ellipse advances (in the direction in which the particle moves) by an amount equal to $\frac{\epsilon}{2C^2}$ of a complete turn. If the inverse-cube force is repulsive, we make ϵ negative, and note that the major axis moves in the opposite sense to that of the particle.

50. Kepler's laws of planetary motion and Newton's law of universal gravitation. The following three laws governing the motion of a planet about the sun were stated by Johannes Kepler (1571–1630).

1. Each planet moves, relative to the sun, in an ellipse having one of its foci at the sun.

2. The areas swept out in equal times by the radius vector to the planet from the sun are equal.

3. Calling the time taken by a planet to describe its elliptical path around the sun its *period*, the cubes of the major axes of the various planetary ellipses are directly proportional to the squares of the corresponding periods.

The first two of these laws were published (in 1609) in Kepler's work on the motions of Mars, and the third (in 1619) in his "De Harmonice Mundi." That Kepler's three laws are equivalent to a single simple law was discovered by Newton. Introducing a system of polar coördinates (r, θ) as in § 49, Kepler's second law states that $r^2\dot{\theta} = C$, indicating that the force exerted by the sun on the planet is central, the center being at the sun. The first law states that

$$u = \frac{1 + e\cos\theta}{l} \quad \text{or} \quad \frac{d^2u}{d\theta^2} + u = \frac{1}{l}.$$

Comparing this with (49.3), we see that the central force is directed toward the sun and is inversely proportional to r^2. Denoting the magnitude of this central force by $\frac{km}{r^2}$, Kepler's third law, as is seen at once from (49.5), states that k is the same for all the planets. Conversely, as we have already seen in § 49, the assumption that there is an attractive central force $\frac{km}{r^2}$ toward the sun acting on each planet, k being the same for all the planets, leads to Kepler's laws $\left(\text{on the assumption that the square of the velocity of each planet is always less than } \frac{2k}{r}\right)$.

In this reduction of Kepler's three empirical laws to a single equivalent statement about the force acting on each planet, the assumption was made that the sun was fixed, that is, that a reference frame whose origin is at the sun and whose axes have fixed directions relative to the fixed stars is an inertial frame. Newton was later led to make his remarkable generalization known as the "law of universal gravitation," according to which any two material particles exert, each upon the other, an absolute force which is directed toward the influencing particle and whose magnitude is inversely proportional to the square of the distance between the particles. Thus the magnitude of the force on a particle m_1, due to the action of a second particle m_2, is $\frac{\lambda_2 m_1}{r^2}$, and the magnitude of the force on m_2 due to the action of m_1 is $\frac{\lambda_1 m_2}{r^2}$, where λ_1 and λ_2 are constants and where r is the distance between the particles. (That the force exerted by each particle is proportional to its mass follows from the principle of the independent action of absolute forces.) Newton's third fundamental law, stating the equality of action and reaction, gives $\lambda_2 m_1 = \lambda_1 m_2$, or $\lambda_1 = \mu m_1$, $\lambda_2 = \mu m_2$, where μ is a constant which is the same for the two particles. The magnitude of the absolute force on m_2 due to the influence of m_1 may, then, be put in the form $\frac{\mu m_1 m_2}{r^2}$. Newton assumed that μ is a universal constant; that is, that its value is the same, no matter where the particles m_1 and m_2 are situated, and no matter what is their chemical constitution or their physical state. In addition, it is assumed to be independent of the velocities of the particles and of any intervening matter. It is known as the gravitational constant; and when the centimeter, gram, and second are employed as the units of length, mass, and time respectively, it has the value 6.66×10^{-8}. The first accurate measurements were made in 1798, by Henry Cavendish (1731–1810).

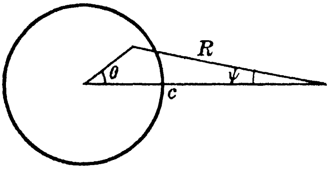


FIG. 28

The sun and the planets are such large distributions of matter that it is important to see how they can, without sensible error, be regarded as particles. Let us first consider the attraction of any spherical distribution of matter upon an external particle, supposing that the density ρ of the matter in the sphere is a function of the distance r from the center. It is apparent, from considerations of symmetry, that the resultant attraction is directed toward the center of the sphere; and its amount is seen to be

$$\mu m \int \frac{\rho r^2 \sin\theta \cos\psi \, dr \, d\theta \, d\phi}{R^2}, \quad (R^2 = r^2 + c^2 - 2\,cr\cos\theta)$$

where m is the mass of the attracted particle, and (r, θ, ϕ) are space polar coördinates whose origin is at the center of the sphere and whose polar axis passes through the particle. Here c is the distance from the center of the sphere to the particle. Upon integration with respect to ϕ, we get $2\,\pi\mu m\int\frac{\rho r^2\sin\theta\cos\psi\,dr\,d\theta}{R^2}$, and we make use of the relation $R^2=r^2+c^2-2\,cr\cos\theta$ to change from the variable θ of integration to the variable R. Since r is constant in the integration with respect to θ, we have $R\,dR=cr\sin\theta\,d\theta$; and, on writing $\cos\psi=\frac{c^2+R^2-r^2}{2\,cR}$, our integral becomes

$$\frac{\pi\mu m}{c^2}\int_0^a\rho r\,dr\int_{c-r}^{c+r}\left(1+\frac{c^2-r^2}{R^2}\right)dR,$$

where a is the radius of the sphere. This integral $=\frac{\mu m}{c^2}\int_0^a 4\,\pi\rho r^2\,dr=\frac{\mu Mm}{c^2}$, where M is the mass of the sphere. Hence a sphere whose mass is distributed in such a way that the density is a function of the distance from the center alone, attracts an external particle as if the entire mass were concentrated at the center of the sphere. From the principle of independent action of absolute forces we infer that such a spherical distribution of mass attracts any distribution of matter external to it as if the total mass of the sphere were concentrated at its center. If the second distribution of matter is also spherical, with its density a function of the distance from the center alone, a second application of this result shows that *the attraction between two such spherical distributions of matter is the same as that between particles of equivalent masses located at their centers.* If, then, we regard the sun and the planets as effectively spherical in shape, with their densities varying merely with the distance from their centers, we may treat them as particles of equivalent masses located at their centers.

Let us denote by M the mass of the sun, and by m the mass of a planet. The origin O of our reference frame, supposed to be located at the center of mass of the sun, has now an acceleration, with respect to an inertial frame, of amount $\frac{\mu m}{r^2}$ directed toward the planet. The *relative* force on the planet is accordingly $\frac{\mu Mm}{r^2}+\frac{\mu m^2}{r^2}$ (see (35.1)). Identifying this with our expression $\frac{km}{r^2}$ of § 49, we have

$$k=\mu(M+m). \tag{50.1}$$

Kepler's third law, which is equivalent to saying that k is the same for all the planets, must therefore be regarded as merely an approximation to Newton's improved statement that the cubes of the major axes are proportional to $M+m$ times the squares of the corresponding periods. The ratio $\frac{m}{M}$ is, however, so small for the various planets that the approximation furnished by Kepler's third law is extremely good. The new statement of the third law may be put in the form

$$\frac{a^3}{T^2} = \frac{\mu(M+m)}{4\,\pi^2}. \tag{50.2}$$

51. Approximate evaluation of the absolute acceleration of the earth's center. The eccentricities of the planetary orbits are so small that these orbits may be regarded as approximately circular (the eccentricity of the earth's orbit is .0168, and that of Mercury, for which the eccentricity is greatest, is .2056). The semimajor axis a of the orbit is, then, known as the mean distance from the sun. For the earth this mean distance is 92.9×10^6 miles; it may be calculated from a knowledge of the sun's parallax, that is, the angle subtended at the center of the sun by two points on the earth whose distance apart is equal to the radius of the earth, or by determining the mean velocity of the earth from the aberration of the stars and the known velocity of light.* The period of the earth in its orbit is approximately 366.25 sidereal days. It follows, from (50.2), that

$$\mu(S+E) = 4.7 \times 10^{21}, \tag{51.1}$$

where S is the mass of the sun, E is the mass of the earth, and the units of length and time adopted are the foot and the second. We may deal similarly with the motion of the moon around the earth, the mean distance a being 238,840 miles, and the period being 27.32 days approximately. We find

$$\mu(E+M) = 1.42 \times 10^{16}, \tag{51.2}$$

where M is the mass of the moon, and the units of length and time are again the foot and the second. Upon dividing (51.1) by (51.2), we have

$$\frac{S}{E} + 1 > \frac{S+E}{E+M} > 3.3 \times 10^5,$$

so that $\frac{S}{E}$ is greater than 3.3×10^5. We may accordingly neglect the mass of the earth in comparison with that of the sun. If we wish to calculate the magnitude $\frac{\mu S}{a^2}$ of the inertial acceleration of the earth due to the attraction of the sun, we may identify this with $\frac{\mu(S+E)}{a^2}$, where $a = 92.9 \times 10^6 \times 5280$ feet. We find in this way, from (51.1), that the magnitude of this acceleration is approximately

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* See F. R. Moulton's "Introduction to Astronomy" (The Macmillan Company, 1919).

$$.019 \text{ feet per second per second.} \tag{51.3}$$

The earth is also attracted by the moon and the other planets, but by far the most predominant terms in its inertial acceleration are those due to the attraction of the sun and the moon. Given a knowledge of the relative mass of the moon and the earth, the inertial acceleration of the earth due to the attraction of the moon may be calculated from (51.2), the result proving to be about six thousandths of that due to the attraction of the sun. (The determination of the relative mass of the moon and the earth involves, however, a discussion of the disturbing action of the sun on the elliptical motion of the moon around the earth.) We see that when the origin of our reference frame is at the center of the earth, the part of the apparent force on any particle which is due to the acceleration of the origin with respect to an inertial frame is very small, since this acceleration is less than .02 feet per second per second. We shall consider this negligible, or, in other words, we shall suppose the center of the earth to be fixed in an inertial reference frame.

It may be well to point out here that even though the acceleration of the earth's center due to the attraction of the moon is less than six thousandths of the acceleration due to the attraction of the sun, the attraction of the moon is the more important when we are considering such questions as the formation of tides etc. When we are dealing with the effect of the attraction of the sun or the moon upon objects on the earth's surface, what comes into play is the *difference* between the acceleration of such an object due to this attraction and the acceleration of the earth's center. If r denotes the distance from the center of the earth to the sun (or moon), and a is the radius of the earth, this difference is of the order of magnitude $C\left[\frac{1}{(r-a)^2} - \frac{1}{r^2}\right]$, or $C\frac{2a}{r^3}$, if $\frac{a}{r}$ is small. In this expression $\frac{C}{r^2}$ denotes the magnitude of the acceleration of the earth's center due to the attraction of the sun (or moon). Since the distance from the earth to the sun is about four hundred times the distance to the moon, the tide-producing effect of the moon proves, by this rough calculation, to be about two and a half times that of the sun.

BIBLIOGRAPHY

In addition to the standard treatises mentioned on page 103, the reader is referred to

MOULTON, F. R. Celestial Mechanics. The Macmillan Company, New York, 1914.

A more advanced treatise, giving an account of the more important applications of the principles of mechanics to astronomy, is

PLUMMER, H. C. Dynamical Astronomy. Cambridge University Press, Cambridge (England), 1918.

EXERCISES

1. Prove that the greatest radial velocity of a particle describing an ellipse about a focus is $\frac{2\,\pi a e(1-e^2)^{-\frac{1}{2}}}{T}$, where $2\,a$ is the major axis, e the eccentricity, and T the periodic time.

2. When a hyperbola is described as a central orbit about a focus, prove that the rate at which areas are described about the center is inversely proportional to the distance from the focus.

3. If the acceleration of a particle is directed to a point F and varies inversely as the square of the distance, prove that there are two directions in which it can be projected from a point P so as to pass through a point Q, and that the velocity of arrival at Q is the same for both. Prove also that the angle between one of the directions of projection and PQ is the same as the angle between the other and PF.

4. Prove that the focal radius and vectorial angle of a particle describing an ellipse of small eccentricity e, at time t after passing the end of the major axis nearer to the focus, are given approximately by the equations

$$r = a(1 - e\cos nt + \tfrac{1}{2}\,e^2 - \tfrac{1}{2}\,e^2\cos 2\,nt),$$
$$\theta = nt + 2\,e\sin nt + \tfrac{5}{4}\,e^2\sin 2\,nt,$$

where $2\,a$ is the major axis and $\frac{2\,\pi}{n}$ is the periodic time. Prove also that if e^2 is neglected, the angular velocity about the other focus is constant.

5. A comet describes a parabola. Show that its velocity perpendicular to the axis of its orbit varies inversely as the radius vector from the sun.

6. A comet of mass m, describing a parabola about the sun, collides with an equal mass m at rest, and the masses move on together. Show that their center of gravity will describe a circle about the sun as a center.

7. The earth's orbit being regarded as a circle, prove that a comet, describing a parabolic orbit in the same plane, cannot remain within the circumference of the earth's orbit longer than $\frac{2}{3\pi}$ of a year.

8. The constituents of a double star describe circles about each other in time T. If they were deprived of velocity and allowed to drop into each other, prove that they would meet after a time $\frac{T}{4(2)^{\frac{1}{2}}}\cdot$

9. A particle moves in a conic, so that the resolved part of the velocity perpendicular to the focal distance is constant. Show that the force tends to the center of the conic.

10. Discuss in some detail the motion of a particle under a central force attracting according to the inverse-cube law.

11. If the law of force is $F = u^2 f(u)$, where $u = \frac{1}{r}$, and the orbit is nearly circular, prove that a first approximation to the path is $u = c[1 + M\cos(p\theta + \alpha)]$, where $p^2 = 1 - \frac{cf'(c)}{f(c)}\cdot$

12. A particle describes a circle about a center of force situated in its plane. Find the law of force and the motion.

CHAPTER VII

THE DYNAMICS OF A SYSTEM OF MATERIAL PARTICLES

52. The external and internal forces of the system. We shall consider a system of n material particles each of which has, at any time $t = t_0$, a known position and velocity relative to any convenient reference frame. Under the gravitational influence of the remaining particles of the system, and the influence of any external absolute forces which may be applied, each particle of the system will possess an acceleration $\mathbf{A}^{(k)}$ relative to the reference frame in use, so that there will be a relative force

$$\mathbf{F}_r^{(k)} = m_k\mathbf{A}^{(k)} \quad (k = 1, 2, \cdots, n)$$

acting on the kth particle, whose mass we denote by m_k. We may analyze $\mathbf{F}_r^{(k)}$ into two parts (see (35.3)):

1. The *absolute* force $\mathbf{F}_{a,i}^{(k)}$, due to the remaining particles of the system. This we shall call the *internal* force acting on the kth particle.

2. The *relative* force $\mathbf{F}_{r,e}^{(k)}$, which the kth particle would experience under the action of the external forces if it were screened from the gravitational attraction of the remaining particles of the system. This we shall call the *external relative* force on the kth particle.

Newton's law of action and reaction postulates that the system of localized vectors $\mathbf{F}_{a,i}^{(k)} (k = 1, 2, \cdots, n)$ is a zero system; that is, that it has a zero resultant $\mathbf{R}$ and a zero moment $\mathbf{G}$ about any point. The equations

$$\mathbf{F}_r^{(k)} = \mathbf{F}_{a,i}^{(k)} + \mathbf{F}_{r,e}^{(k)} \quad (k = 1, 2, \cdots, n) \qquad (52.1)$$

then tell us that the system of localized vectors

$$\mathbf{F}_r^{(k)} \quad (k = 1, 2, \cdots, n)$$

is equivalent to the system of localized vectors $\mathbf{F}_{r,e}^{(k)}$. Since $\mathbf{F}_r^{(k)} = m_k\ddot{\mathbf{r}}_k$, where $\mathbf{r}_k$ is the vector from the origin O of our reference frame to the kth particle, the system $\mathbf{F}_r^{(k)}$ is the time derivative of the system of localized vectors $m_k\dot{\mathbf{r}}_k$ with respect to the reference frame in use. This system may be called the *momentum system.* Its resultant

$$\mathbf{R} = \sum_{k=1}^{k=n} m_k \dot{\mathbf{r}}_k \tag{52.2}$$

is the total relative linear momentum of the system of n particles, and its vector moment $\mathbf{G}$ about the origin

$$\mathbf{G} = \sum_{k=1}^{k=n} [\mathbf{r}_k \, . \, m_k \dot{\mathbf{r}}_k] \tag{52.3}$$

is the total relative angular momentum of the system of n particles about the origin. We have, then, the following fundamental result:

The time derivative of the momentum system of localized vectors is equivalent to the external-applied-relative-force system of localized vectors.

If we denote the center of mass of the n particles by G, the vector $\overrightarrow{OG} = \mathbf{r}_G$ is defined by the equation

$$M\mathbf{r}_G = \sum_{k=1}^{k=n} m_k \mathbf{r}_k, \quad \text{where} \quad M = \sum_{k=1}^{k=n} m_k. \tag{52.4}$$

The resultant of the momentum system of localized vectors is thus $\mathbf{R} = M\dot{\mathbf{r}}_G$. Hence we have

$\dfrac{d}{dt}(M\dot{\mathbf{r}}_G) =$ the resultant vector of the external-applied-force system.

In other words, the center of mass of the system of n particles moves as if the total mass of the system were concentrated there and all the external applied forces acted through it. It is this result which makes so useful the results of *particle dynamics.* A material particle is a fiction, but the center of mass of a system which can be imagined as built up of a number of material particles moves as if it were a material particle itself.

If we happen to be using a non-inertial reference frame which has no rotation with respect to an inertial frame, the part of the relative external force on the kth particle which is due to the acceleration of the reference frame with respect to an inertial frame is $-m_k\mathbf{A}$, where $\mathbf{A}$ is the acceleration of any point of the framework relative to the inertial frame. This localized-vector system consists of parallel vectors and is equivalent to a single localized vector through the center of mass of the system of particles (see § 20). Hence if we take moments about the center of mass, we may neglect the centrifugal forces due to the non-inertial character of the frame. In other words,

The time rate of change of the angular momentum of the system about its center of mass is equal to the moment of the external applied absolute forces about this center.

Thus we may say that the motion of the system of particles about its center of mass is the same as if the center of mass were fixed in an inertial frame. Of course, this result holds only for a non-inertial frame without rotation.

From the results just obtained we see that it is convenient to divide the discussion of the motion of a system of material particles into two parts: (1) the motion of the center of mass (a problem in particle dynamics); (2) the motion of the system relative to its center of mass.

A particularly simple, yet important, case occurs when there are no external applied forces. We are then dealing with a system of material particles moving under the influence of the internal forces of the system alone. The center of mass will remain at rest in an inertial frame, and the angular momentum of the system about its center of mass will not vary with the time. Regarding the solar system as an example of such a dynamic system *left to itself,* we have Laplace's invariable plane, that is, a plane through the center of mass of the system (practically the center of the sun) perpendicular to the representative segment of the angular momentum **G** of the solar system about its center of mass.

The various bodies of the solar system are here regarded as particles. If they are regarded as spheres whose strata of equal density are concentric spheres, the force on each body due to the attraction of the others passes through its center. Hence its angular momentum about its center is constant. If we take this angular momentum into account, we obtain a second invariable plane known as the dynamic invariable plane. (Laplace's invariable plane is known as the astronomical invariable plane.) In each case we are calculating the total angular momentum of the system about its center of mass. In the case of the astronomical invariable plane the angular momentum due to the rotation of each planet is neglected in the calculation.

53. The energy of a material dynamic system. If we multiply the equation

$$m_k\ddot{\mathbf{r}} = \mathbf{F}_r^{(k)} = \mathbf{F}_{r,e}^{(k)} + \mathbf{F}_{a,i}^{(k)}$$

scalarly by $\dot{\mathbf{r}}_k$, and sum the resulting equation over the n particles of the system, we obtain

$$\frac{dT}{dt} = \sum_{k=1}^{k=n}[(\mathbf{F}_{r,e}^{(k)}.\dot{\mathbf{r}}_k) + (\mathbf{F}_{a,i}^{(k)}.\dot{\mathbf{r}}_k)], \qquad (53.1)$$

where $T = \frac{1}{2}\,\Sigma_k m_k(\dot{\mathbf{r}}_k.\dot{\mathbf{r}}_k) = \frac{1}{2}\,\Sigma_k m_k v_k^2$ is the relative *kinetic energy* of the system. On the right-hand side of (53.1) we have the *power*, or *activity*, of the external relative forces and the internal absolute forces. We may state our equation in words as follows:

The time rate of increase of the kinetic energy of the system is equal to the rate at which the external relative forces and the internal absolute forces do work on the system.

The internal force $\mathbf{F}_{a,i}^{(k)}$ on the kth particle is the sum of the absolute forces due to the other particles of the system. We know, from Newton's law of action and reaction, that the force on the kth particle due to the jth particle acts along their join and is equal and opposite to the force on the jth particle due to the kth particle. We may, accordingly, write the force on the kth particle due to the jth particle in the form $\dfrac{f_{kj}(\mathbf{r}_k - \mathbf{r}_j)}{r_{kj}}$, where $f_{kj} = f_{jk}$ and where r_{kj} is the distance between the two particles. Hence the rate at which the internal forces do work on the system is

$$\sum_{k=1}^{n}\sum_{j=1}^{n} f_{kj}\left(\frac{\mathbf{r}_k - \mathbf{r}_j}{r_{kj}}\cdot\dot{\mathbf{r}}_k\right). \quad (j \neq k)$$

Since $f_{kj} = f_{jk}$, this may be written in the form

$$\sum_{k,j\ 1}^{n} f_{kj}\,\frac{(\mathbf{r}_k - \mathbf{r}_j)(\dot{\mathbf{r}}_k - \dot{\mathbf{r}}_j)}{r_{kj}}. \quad (j > k) \qquad (53.2)$$

Using the relation $\qquad (\mathbf{r}_k - \mathbf{r}_j)^2 = r_{kj}^2,$

we obtain $\qquad (\mathbf{r}_k - \mathbf{r}_j)(\dot{\mathbf{r}}_k - \dot{\mathbf{r}}_j) = r_{kj}\,\dot{r}_{kj},$

so that the activity of the internal forces is

$$\sum_{k,j\ 1}^{n} f_{kj}\,\dot{r}_{kj}. \quad (j > k) \qquad (53.3)$$

If all the distances r_{kj} are constant, the system of n particles is said to form a rigid dynamic system; and, from (53.3), we have the result that *the power of the absolute internal forces of a rigid dynamic system is zero.*

We shall now make the assumption that each of the quantities f_{kj} is a function of the corresponding distance r_{kj} alone. Upon introducing the function V_i of the mutual distances r_{kj}, as defined by

$$V_i = -\sum_{k,j\atop 1}^{n} \int^{r_{kj}} f_{kj}\,dr_{kj}, \quad (j > k)$$

we have

$$\frac{\partial V_i}{\partial r_{kj}} = -f_{kj},$$

so that

$$\dot{V}_i = -\sum_{k,j\atop 1}^{n} f_{kj}\,\dot{r}_{kj}. \quad (j > k)$$

In other words, the rate at which the internal forces do work on the system is $-\dot{V}_i$; and $-V_i$ may be regarded as measuring the work done on the system by the internal forces, measured from the configuration at any convenient initial time $t = t_0$. Our equation (53.1) may now be rewritten in the form

$$\frac{d}{dt}(T + V_i) = \sum_{k=1}^{n} (\mathbf{F}_{r,e}^{(k)} \cdot \dot{\mathbf{r}}_k), \tag{53.4}$$

where V_i is called the internal potential energy of the system. This gives us the so-called *energy principle*:

The rate of increase of the sum of the kinetic and the potential energy of the system is equal to the rate at which the external relative forces do work on the system.

An especially important case arises when the external-force system is a gradient system. By this is meant that there exists a scalar function V_e of the vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ such that

$$\mathbf{F}_{r,e}^{(k)} = -\mathbf{grad}_k V_e;$$

that is,

$$X_{r,e}^{(k)} = -\frac{\partial V_e}{\partial x_k}, \quad Y_{r,e}^{(k)} = -\frac{\partial V_e}{\partial y_k}, \quad Z_{r,e}^{(k)} = -\frac{\partial V_e}{\partial z_k},$$

where $(X_{r,e}^{(k)}, Y_{r,e}^{(k)}, Z_{r,e}^{(k)})$ are the components, in the reference frame in use, of the relative-force vector $\mathbf{F}_{r,e}^{(k)}$. In this case we have $\sum_{k=1}^{n}(\mathbf{F}_{r,e}^{(k)}.\dot{\mathbf{r}}_k) = -\dot{V}_e$, which shows that the rate at which the external relative forces do work on the system is $-\dot{V}_e$; so that our result (53.4) takes the form

$$\frac{d}{dt}(T + V_i + V_e) = 0, \tag{53.4 a}$$

or

$$T + V_i + V_e = \text{constant}. \tag{53.5}$$

Calling V_e the external potential energy of the system, the sum $V = V_i + V_e$ is called the total potential energy of the system; and the equation

$$T + V = \text{constant} \tag{53.5 a}$$

says that the total energy of the system, both kinetic and potential, is constant. This result is known as the energy integral of the equations of motion of the system. The external forces are said to be conservative, the energy integral being referred to as the principle of the conservation of energy. Systems possessing an energy integral may be said to be frictionless. Frictional forces do not have a potential, since the work done in a path between two points depends upon the path. In such a case, equation (53.4) would read as follows:

The rate of decrease in energy of the system is equal to the rate at which work is done against the external relative forces.

This theorem has important applications in thermodynamics.

54. The kinetic energy and angular momentum of a rigid dynamic system. When we are dealing with a rigid dynamic system, it is convenient to introduce as a moving reference frame one which is attached to the system. The relative velocity of any particle of the system being then zero, the velocity $\mathbf{v}_k$ of the kth particle is, by (27.2),

$$\mathbf{v}_k = \mathbf{v}_0 + [\boldsymbol{\omega}.\mathbf{r}_k], \tag{54.1}$$

where $\mathbf{v}_0$ is the velocity of the origin O, and $\boldsymbol{\omega}$ is the angular velocity, relative to the frame with respect to which we wish to calculate the kinetic energy and the angular momentum of the system, of the frame which is attached to the rigid system. It follows from (54.1), on using (3.11) and (3.19), that

$$v_k^2 = v_0^2 + (2\,\boldsymbol{\omega}.[\mathbf{r}_k.\mathbf{v}_0]) + \omega^2 r_k^2 - (\mathbf{r}_k.\boldsymbol{\omega})^2.$$

Denoting the vector $\overrightarrow{OG}$, where G is the center of mass, by $\mathbf{r}_G$, we have for the absolute kinetic energy T the expression

$$T = \tfrac{1}{2}\sum_{k=1}^{n} m_k v_k^2 = \tfrac{1}{2}\, M v_0^2 + M(\boldsymbol{\omega}.[\mathbf{r}_G.\mathbf{v}_0]) \\ + \tfrac{1}{2}\sum_{k=1}^{n} m_k[\omega^2 r_k^2 - (\mathbf{r}_k.\boldsymbol{\omega})^2], \quad (54.2)$$

where $M = \sum_{k=1}^{n} m_k$ is the total mass of the system. If we denote by d_k the distance of the particle m_k from the representative segment through O of the rotation vector $\boldsymbol{\omega}$, we have

$$d_k^2 = r_k^2 - \frac{(\mathbf{r}_k.\boldsymbol{\omega})^2}{\omega^2};$$

and we may write

$$T = \tfrac{1}{2}\, M v_0^2 + M(\boldsymbol{\omega}.[\mathbf{r}_G.\mathbf{v}_0]) + \tfrac{1}{2}\, I\omega^2, \qquad (54.3)$$

where $I = \sum_{k=1}^{n} m_k d_k^2$ is known as the moment of inertia of the system of n particles about the axis through O of the rotation vector $\boldsymbol{\omega}$.

The angular momentum of the system of n particles about the origin O is

$$\Sigma_k[\mathbf{r}_k.m_k\mathbf{v}_k],$$

and this is, by (54.1),

$$\mathbf{G} = M[\mathbf{r}_G.\mathbf{v}_0] + \sum_{k=1}^{n} m_k[r_k^2\boldsymbol{\omega} - (\mathbf{r}_k.\boldsymbol{\omega})\mathbf{r}_k]. \qquad (54.4)$$

This expression may be readily remembered in the symbolic form

$$\mathbf{G} = \frac{\partial T}{\partial \boldsymbol{\omega}},$$

that is, $$G_x = \frac{\partial T}{\partial \omega_x}, \quad G_y = \frac{\partial T}{\partial \omega_y}, \quad G_z = \frac{\partial T}{\partial \omega_z}, \qquad (54.5)$$

which follows upon comparison of (54.2) and (54.4). It is apparent that a considerable simplification is introduced if we take the origin of the frame, attached to the rigid system of particles, at G, the mean center of the system. Using V to denote the magnitude of the velocity of the center of mass, we have, for the absolute kinetic energy T and the absolute angular-momentum vector $\mathbf{G}$ about the center of mass,

$$T = \tfrac{1}{2}\, MV^2 + \tfrac{1}{2}\, I\omega^2, \qquad (54.3\,a)$$

$$\mathbf{G} = \frac{\partial T}{\partial \boldsymbol{\omega}}. \qquad (54.5\,a)$$

In using this last symbolic equation it must not be forgotten that I is a function of $\boldsymbol{\omega}$, since if the direction of the vector $\boldsymbol{\omega}$ changes, I takes a new value.

55. The moment of inertia of a dynamic system. The moment of inertia $I = \Sigma_k m_k d_k^2$ of the system of n particles about any line through O is given by

$$I = \Sigma_k m_k [r_k^2 - (\mathbf{r}_k . \boldsymbol{\nu})^2],$$

where $\boldsymbol{\nu}$ is the unit vector whose representative segment lies along the line through O about which the moment is being taken. Let us change our origin to a new point O' and calculate the moment of inertia about a parallel line through O'. We have $\mathbf{r}_k = \mathbf{r}' + \mathbf{r}'_k$, where $\mathbf{r}'$ and $\mathbf{r}'_k$ are the nonlocalized vectors which have $\overrightarrow{OO'}$ and $\overrightarrow{O'P_k}$, respectively, as representative segments. From this it follows that

$$I = I' + M[r'^2 - (\mathbf{r}'.\boldsymbol{\nu})^2] + 2(\Sigma_k m_k \mathbf{r}'_k . \mathbf{r}' - (\mathbf{r}'.\boldsymbol{\nu})\boldsymbol{\nu}). \quad (55.1)$$

When the new origin O' is the center of mass G of the system, $\Sigma_k m_k \mathbf{r}'_k = \mathbf{0}$, and (55.1) takes the form

$$I = I' + M[r'^2 - (\mathbf{r}'.\boldsymbol{\nu})^2]. \quad (55.2)$$

This is known as the parallel-axis theorem. It states that the moment of inertia about any straight line may be found by adding to the moment of inertia about a parallel straight line through the center of mass the product of the total mass by the square of the distance between the two parallel lines.

Let us now keep the origin O fixed and consider the moment of inertia $I = \Sigma_k m_k [r_k^2 - (\mathbf{r}_k.\boldsymbol{\nu})^2]$ as a function of the unit vector $\boldsymbol{\nu}$. To do this we write it out in the form

$$I = A\nu_x^2 + B\nu_y^2 + C\nu_z^2 - 2F\nu_y\nu_z - 2G\nu_z\nu_x - 2H\nu_x\nu_y,$$

where $A = \Sigma_k m_k(y_k^2 + z_k^2)$, etc., $F = \Sigma_k m_k y_k z_k$, etc.

Here A, B, and C are the moments of inertia about the coördinate axes, that is, the values of I when ν has the presentations (1, 0, 0), (0, 1, 0), and (0, 0, 1) respectively. F, G, and H are known as the products of inertia about the three coördinate planes. Since I is a scalar quantity, it follows, from §§ 8 and 9, that $(A, B, C, -F, -G, -H)$ are the six distinct components, in the coördinate frame in use, of a symmetric tensor of the second rank. Furthermore, since I, from its very definition, cannot be negative, the representative quadric surfaces of this tensor are ellipsoids. The particular representative quadric

$$Ax^2 + By^2 + Cz^2 - 2\,Fyz - 2\,Gzx - 2\,Hxy = 1 \qquad (55.3)$$

is known as the momental ellipsoid of the system at the point O. It has the property that the radius vector r along $\mathbf{v}$ has the length $\sqrt{\frac{1}{I}}$. The axes of the momental ellipsoid are known as the principal axes of the system at the point O.

The formulas for the kinetic energy and the angular momentum of the system about O, in terms of the components of the moment-of-inertia tensor, follow from the fact that

$$I\omega^2 = A\omega_x{}^2 + B\omega_y{}^2 + C\omega_z{}^2 - 2\,F\omega_y\omega_z - 2\,G\omega_z\omega_x - 2\,H\omega_x\omega_y.$$

Taking our origin at the center of mass G, we have

$$\begin{aligned} T = \tfrac{1}{2}\,MV^2 + \tfrac{1}{2}(A\omega_x{}^2 &+ B\omega_y{}^2 \\ &+ C\omega_z{}^2 - 2\,F\omega_y\omega_z - 2\,G\omega_z\omega_x - 2\,H\omega_x\omega_y), \\ G_x &= \frac{\partial T}{\partial \omega_x} = A\omega_x - H\omega_y - G\omega_z, \\ G_y &= \frac{\partial T}{\partial \omega_y} = -\,H\omega_x + B\omega_y - F\omega_z, \qquad (55.4) \\ G_z &= \frac{\partial T}{\partial \omega_z} = -\,G\omega_x - F\omega_y + C\omega_z. \end{aligned}$$

If the dynamic system has a fixed point, we may put the origin there. We then have

$$\begin{aligned} T &= \tfrac{1}{2}\,I\omega^2 \\ &= \tfrac{1}{2}(A\omega_x{}^2 + B\omega_y{}^2 + C\omega_z{}^2 - 2\,F\omega_y\omega_z - 2\,G\omega_z\omega_x - 2\,H\omega_x\omega_y). \end{aligned}$$

[See (54.3)]

From this we derive the same expressions as in (55.4) for the components of $\mathbf{G}$. The resolved part of $\mathbf{G}$ along $\boldsymbol{\omega}$ is $\frac{2\,T}{\omega}$ or $I\omega$ in either case. Expressed in words, the angular momentum about an axis of a rigid system, whose motion is one of pure rotation about this axis, is $I\omega$, where I is the moment of inertia of the system about the same axis. The same result holds, even when the motion is not one of pure rotation about an axis, provided the axis, about which the angular momentum is being calculated, passes through the center of mass of the system. Thus *the center of mass plays the rôle of a fixed point*, as far as this theorem goes.

When the reference frame is not specialized in any way, the corresponding results are

$$\begin{aligned} T=\tfrac{1}{2}\, Mv_0{}^2+M(\boldsymbol{\omega}.[\mathbf{r}_G.\mathbf{v}_0]) &+\tfrac{1}{2}(A\omega_x{}^2+B\omega_y{}^2 \\ &+C\omega_z{}^2-2\,F\omega_y\omega_z-2\,G\omega_z\omega_x-2\,H\omega_x\omega_y), \end{aligned}$$

$$\begin{aligned} G_x &= M(\beta\zeta-\gamma\eta)+A\omega_x-H\omega_y-G\omega_z,\\ G_y &= M(\gamma\xi-\alpha\zeta)-H\omega_x+B\omega_y-F\omega_z, \qquad (55.4\,a)\\ G_z &= M(\alpha\eta-\beta\xi)-G\omega_x-F\omega_y+C\omega_z, \end{aligned}$$

where (α, β, γ) are the coördinates of the center of mass and where (ξ, η, ζ) are the components of the velocity of the origin.

56. Principal axes of inertia; spherical and circular points. It follows, from the equations of definition

$$A=\Sigma_k m_k(y_k{}^2+z_k{}^2), \text{ etc.,} \qquad F=\Sigma_k m_k y_k z_k, \text{ etc.,}$$

that if we introduce a parallel reference frame whose origin O' has coördinates (a, b, c) in the old frame, the new moments and products of inertia are

$$\begin{aligned} A' &= A-2\,M(b\beta+c\gamma)+M(b^2+c^2), \text{ etc.}\\ F' &= F-M(b\gamma+c\beta)+Mbc, \text{ etc.,} \end{aligned} \qquad (56.1)$$

(α, β, γ) being the coördinates of the center of mass of the system. If the original origin is at the center of mass, $(\alpha, \beta, \gamma)=(0, 0, 0)$, and we have

$$A'=A+M(b^2+c^2), \text{ etc.,} \qquad F'=F+Mbc, \text{ etc.} \qquad (56.2)$$

Let us now seek the condition that a straight line be a principal axis at one of its points. Taking the line as the z-axis of our reference frame, it is a principal axis at $(0, 0, c)$; and we find, on equating F' and G' to zero in (56.1), that

$$F-Mc\beta=0, \quad G-Mc\alpha=0, \quad \text{or} \quad \frac{F}{\beta}=\frac{G}{\alpha}=Mc.$$

If a line is a principal axis at more than one of its points, the value of c, as found here, must be indeterminate, and the line is a principal axis at all its points. In this exceptional case we have $\alpha=0$, $\beta=0$, $F=0$, $G=0$, so that the line passes through the center of mass of the system and is one of the principal axes at the center of mass. Hence every line not passing through the center of mass is a principal axis at some point; but there are, in general, only three lines — namely, the three principal axes at the center of mass — which are principal axes at all their points. Lines through the center of mass which are not principal axes at the center of mass are not principal axes at any point.

It follows conversely, from (56.2), that if a line through the center of mass is a principal axis at one of its points, it must be one of the principal axes of inertia at the center of mass. Let us see if a dynamic system in general has any spherical points, that is, points for which the momental ellipsoid is a sphere. Any line through a spherical point is a principal axis of inertia at the spherical point. In particular, the join of the center of mass to the spherical point is a principal axis at this point. Hence the spherical point must lie on one of the principal axes at the center of mass. Taking the principal axis on which it lies as the z-axis, the origin being at the center of mass, and equating the A', B', and C' of (56.2), we find

$$C = A + Mc^2 = B + Mc^2.$$

Hence $A = B$; and if this condition is satisfied, there are two spherical points on the principal axis of inertia of greatest moment at the center of mass, their common distance from the center of mass being $\sqrt{\dfrac{C - A}{M}}$.

When the n material particles of the system all lie in the same plane, the moment of inertia about any line in the plane through a point O is equal to the inverse square of the radius vector of the momental ellipse at O. This is the section of the momental ellipsoid at O by the plane of the particles. At each point of the plane there are two principal axes of inertia in the plane, the third principal axis of inertia being the perpendicular at O to the plane of the particles. The equation of the momental ellipse, when referred to the principal axes, is $Ax^2 + By^2 = 1$. When we take a system of parallel axes through (a, b), the new moments and products of inertia are connected with the old ones by the equations

$$\begin{gathered} A' = A - 2\,Mb\beta + Mb^2, \quad B' = B - 2\,Ma\alpha + Ma^2, \\ H' = H - M(b\alpha + a\beta) + Mab, \end{gathered} \tag{56.3}$$

where (α, β) are the coördinates of the center of mass. When the origin is at the center of mass, the formulas are

$$A' = A + Mb^2, \quad B' = B + Ma^2, \quad H' = H + Mab. \tag{56.4}$$

Each line in the plane which does not pass through the center of mass is a principal axis at one of its points. For on taking it as the x-axis, we have, on writing $H' = 0$, $b = 0$, in (56.3), $a = \dfrac{H}{M\beta}$. If a line through the center of mass is a principal axis at one of its points, it is a principal axis at all its points, and so it must be one of the two principal axes at the center of mass. From this we see that a circular point, that is, a point whose momental ellipse is a circle, must lie on one of the two principal axes through the center of mass. Taking it as the x-axis and equating A' and B' in (56.4), we have $A = B + Ma^2$. Hence there are two circular points situated on the principal axis of greater moment through the center of mass at a common distance $\sqrt{\frac{A-B}{M}}$ from the center of mass. Since the moment of inertia about any line in the plane of the particles through either circular point is equal to the moment about their join (that is, since it is equal to A), the moments of inertia about the two lines joining any point in the plane of the particles to the circular points are each equal to A. Hence the principal axes at any point bisect the angles between the two lines joining that point to the two circular points. In other words, they are the tangent and normal to the ellipse through the point which has the two circular points as foci. For this reason the two circular points are known also as *foci of inertia.* It follows at once, on using the parallel-axis theorem, that if d_1 and d_2 denote the distances, from the circular points, of any straight line in the plane of the particles, the moment of inertia about the straight line is $A \pm Md_1d_2$. The positive sign is used when the two circular points are on the same side of the straight line, and the negative sign when the two circular points are on opposite sides of the line.* The straight lines in the plane of the particle about which the moment of inertia has a given value $I > A$ are accordingly tangent to an ellipse which has the circular points as foci and whose minor axis b is given by $b^2 = d_1d_2 = \frac{I-A}{M}$. The equation of the ellipse is

$$\frac{x^2}{I-B} + \frac{y^2}{I-A} = \frac{1}{M}. \tag{56.5}$$

Similarly, the straight lines in the plane of the particles about which the moment of inertia has a constant value $I < A$ are tangent to the hyperbola

$$\frac{x^2}{I-B} - \frac{y^2}{A-I} = \frac{1}{M}. \tag{56.6}$$

(I is necessarily $\geqq B$, since of all lines in the plane of the particles the principal axis of lesser moment through the center of mass is the line about which the moment of inertia is least.)

57. Extension of the concept of a dynamic system. We have so far regarded our dynamic system as made up of a finite number n of particles. It is easy to extend this so as to include the case of a continuous distribution of matter. Thus we may regard a volume distribution as the limit of a great number of small particles closely packed together. Denoting by Δm the mass of a small element Δv of the volume, the density ρ of the volume distribution of matter is defined by the equation

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* If p denotes the distance of the straight line from the center of mass, and d_1 its distance from the nearer of the two foci, the moment of inertia about a parallel line through the center of mass is $A - M(p \mp d_1)^2$. The moment about the line is $A - M(p \mp d_1)^2 + Mp^2$. This gives the result stated, since $d_2 \pm d_1 = 2p$.

$$\rho = \underset{\Delta v \to 0}{\mathrm{Lt}} \frac{\Delta m}{\Delta v}.$$

The total mass M of the body is then given by $M = \int \rho \, dv$, the triple integral being extended over the volume distribution of matter. If $\mathbf{r}$ is the vector $\overrightarrow{OP}$ from the origin O to the point P of the body where the density is ρ, the vector $\overrightarrow{OG}$ from O to the center of mass is given by the equation

$$M\mathbf{r}_G = \int \mathbf{r} \, dm = \int \rho \mathbf{r} \, dv.$$

Similarly, the moments and products of inertia of the body about the coördinate axes and planes are given by

$$A = \int (y^2 + z^2) dm = \int \rho(y^2 + z^2) dv, \text{ etc.},$$

$$F = \int yz \, dm = \int \rho yz \, dv, \text{ etc.}$$

If we have a surface distribution of matter, we proceed in exactly the same way, introducing the concept of a superficial density σ defined by $\sigma = \underset{\Delta S \to 0}{\mathrm{Lt}} \frac{\Delta m}{\Delta S}$, where Δm is the mass of the element of area ΔS. If we have a linear distribution of matter, we introduce the idea of a linear density λ defined by the equation $\lambda = \underset{\Delta s \to 0}{\mathrm{Lt}} \frac{\Delta m}{\Delta s}$, where Δm is the mass of the element of arc Δs of the curve along which the matter is distributed.

EXERCISES

1. Show that the moment of inertia of a uniform straight rod of length $2a$ about a straight line perpendicular to it through its center of mass is $\frac{Ma^2}{3}$.

2. Show that the moment of inertia of a uniform rectangle of sides $2a$ and $2b$ about a line perpendicular to it through its center is $\frac{M(a^2 + b^2)}{3}$.

3. Show that the moment of inertia of a uniform elliptical area whose major and minor axes are 2 a and 2 b, respectively, about its major axis is $\frac{Mb^2}{4}$; about its minor axis is $\frac{Ma^2}{4}$.

4. Show that the moment of inertia of a uniform ellipsoid whose semiaxes are a, b, c about the first axis is $\frac{M(b^2+c^2)}{5}$. From this, or independently, show that the moment of inertia of a uniform sphere of radius a about a diameter is $\frac{2\,Ma^2}{5}$.

These results may all be collected into a form of statement known as *Routh's rule*:

Moment of inertia about an axis of symmetry

$$= \frac{M(\text{sum of squares of perpendicular semiaxes})}{3,\ 4,\ \text{or } 5}.$$

(The denominator is to be 3, 4, or 5 according as the body is rectangular, elliptical, or ellipsoidal.)

58. Euler's equations; dynamic significance of the principal axes of inertia. Since the time derivative of the angular-momentum vector **G** is equal to the moment of the external applied forces (§ 52), we have, on using (26.5), the three equations

$$\begin{aligned}\frac{dG_x}{dt} + \omega_y G_z - \omega_z G_y &= L,\\ \frac{dG_y}{dt} + \omega_z G_x - \omega_x G_z &= M,\\ \frac{dG_z}{dt} + \omega_x G_y - \omega_y G_x &= N,\end{aligned} \tag{58.1}$$

where (L, M, N) are the components of the moment of the external applied forces about O. If the origin of our reference frame (which moves with the body) is either at rest in the frame with respect to which (L, M, N) are calculated, or is located at the center of mass of the system, we may write

$$G_x = A\omega_x - H\omega_y - G\omega_z, \text{ etc.}$$

Otherwise we have, in the expression for **G**, the additional vector $M[\mathbf{r}_G \cdot \mathbf{v}_0]$ (see (55.4 a)). If we take the principal axes of inertia at the fixed point (or center of mass) as forming our moving reference frame, we have $G_x = A\omega_x$, etc., and our equations (58.1) take the form

$$\begin{aligned} A\frac{d\omega_x}{dt} - (B - C)\omega_y\omega_z &= L, \\ B\frac{d\omega_y}{dt} - (C - A)\omega_z\omega_x &= M, \\ C\frac{d\omega_z}{dt} - (A - B)\omega_x\omega_y &= N. \end{aligned} \tag{58.2}$$

In this form they are known as Euler's equations. Let us now suppose that we have a body, containing a fixed point, turning about an axis fixed in it. Choosing this axis as the z-axis, we have $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$, so that $G_x = -G\omega$, $G_y = -F\omega$, $G_z = C\omega$; and equations (58.1) give

$$\begin{aligned} -G\frac{d\omega}{dt} + F\omega^2 &= L, \\ -F\frac{d\omega}{dt} - G\omega^2 &= M, \\ C\frac{d\omega}{dt} &= N. \end{aligned} \tag{58.3}$$

It is therefore necessary, in general, to apply moments, by means of bearings etc., to make a body rotate about an axis fixed in it. In order to see what conditions are necessary in order that a body may rotate freely about an axis fixed in it, we put $L = 0$, $M = 0$, $N = 0$, in (58.3); and we find

$$\frac{d\omega}{dt} = 0, \quad F = 0, \quad G = 0.$$

In other words, the body must be rotating with constant angular velocity, and the fixed axis in the body about which it is rotating must be a principal axis of inertia. Conversely, if a body with a fixed point is set rotating about a principal axis through the point, it will continue to rotate about this principal axis with uniform angular velocity unless a moment is applied about the fixed point. For this reason the principal axes of inertia at a point are often referred to as *permanent* or *free axes of rotation.* (As has been remarked above, the center of mass here plays the rôle of a fixed point.) Equations (58.3) give the moment

(L, M, N) of the applied-force system about O. The resultant $\mathbf{R}$ of this system is given (see § 52) by

$$m\ddot{\mathbf{r}}_G = \mathbf{R},$$

where m is the mass of the body. The magnitude of $\mathbf{R}$ is therefore $mr_G\omega^2$. To avoid dangerous thrusts on the bearings of a rapidly spinning body the axis of rotation should pass through the center of mass.

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Of the texts mentioned on page 103, Routh's "Elementary Rigid Dynamics" gives the best treatment of moments and axes of inertia. For the distinction between the internal and external forces acting on a material system we may refer to

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EXERCISES

1. Show that the foci of inertia for a uniform elliptic lamina lie on the minor axis at distances from the center each equal to $\frac{1}{2}\, ae$.

2. Discuss the motion of a particle on a surface of rotation when the potential energy is a function of the distance from the axis of rotation alone. Apply in particular to the case of the motion of a particle on a right circular cone.

3. A particle of mass m is placed in a smooth linear tube which can rotate in a vertical plane about its middle point. The system starts from rest when the tube is horizontal. If θ is the angle the tube makes with the vertical when its angular velocity is a maximum and equal to ω, prove that $4(mr^2 + Mk^2)\omega^4 - 8\, mgr\omega^2 \cos\theta + mg^2 \sin^2\theta = 0$, where Mk^2 is the moment of inertia of the tube about its center, and r is the distance of the particle from the center of the tube.

4. A uniform rod moves with its ends on a smooth circular wire fixed in a vertical plane. Prove that if it subtends an angle of 120 degrees at the center, the length of the simple equivalent pendulum is equal to the radius of the circle.

5. A circular cylinder which has a radius a, and whose moment of inertia about its axis is Mk^2, rolls inside a fixed horizontal cylinder of radius b. Prove that the plane through the axes moves like a simple pendulum of length $(b - a)\dfrac{a^2 + k^2}{a^2}$.

6. The line of hinges of a door makes an angle α with the vertical, and the door swings about its position of equilibrium. Show that the motion is the same as that of a simple pendulum. Find the length of this pendulum.

7. Two bicyclists, riding exactly similar machines and starting with equal velocities at the top, coast down a hill. Neglecting the forces of friction and the resistance of the air, show that the heavier rider will reach the bottom first.

8. A straight piece of uniform wire is stood vertically on end and allowed to fall over. With what velocity does it strike the ground?

9. Find the moment of inertia of a circular disk (1) about an axis through its center perpendicular to its plane and (2) about a diameter.

10. Deduce the position of the two principal axes of inertia in the plane of a rectangular plate having their origin at one of its corners.

11. Find the moment of inertia of a square lamina about a diagonal.

12. Two masses, M and m, suspended from a wheel and axle of radii a, b, do not balance. Show that the acceleration of M is $\frac{(Ma - mb)ag}{Ma^2 + mb^2 + I}$, where I is the moment of inertia of the machine about its axis.

13. Find the moment of inertia of a right circular cone about its altitude and about a diameter of its base.

CHAPTER VIII

IMPULSIVE FORCES

59. Impulse. Upon integration of the fundamental equation $m\ddot{\mathbf{r}} = \mathbf{F}_r$, which governs the motion of a single material particle, with respect to the time t, we obtain

$$m(\dot{\mathbf{r}} - \dot{\mathbf{r}}_0) = \int_{t_0}^{t} \mathbf{F}_r \, dt. \tag{59.1}$$

The vector on the right is known as the *impulse* of the force $\mathbf{F}_r$ on the particle during the time interval t_0 to t. Denoting this impulse by $\mathbf{I}(t)$, we may rewrite (59.1) in the form

$$m(\dot{\mathbf{r}} - \dot{\mathbf{r}}_0) = \mathbf{I}(t), \tag{59.1 a}$$

so that the change of (linear) momentum of the particle in the time interval t_0 to t is equal to the impulse of the force $\mathbf{F}_r$ acting on the particle during that interval. The impulse $\mathbf{I}(t)$ will approach zero as the time interval $t - t_0 = \Delta t$ tends to zero, unless the force $\mathbf{F}_r \rightarrow \infty$ as $t \rightarrow t_0$. We shall suppose that as $t \rightarrow t_0$, $\mathbf{F}_r \rightarrow \infty$ in the order of magnitude $\frac{1}{\Delta t}$; so that the impulse $\mathbf{I}(t)$ tends to a definite limiting value $\mathbf{I}$ as $t \rightarrow t_0$. The force $\mathbf{F}_r$ is then called an impulsive force, and the limit $\mathbf{I}$ of $\mathbf{I}(t)$ as $t \rightarrow t_0$ is known as the impulse of the impulsive force $\mathbf{F}_r$ acting at the instant $t = t_0$. An impulsive force causes a sudden change in the velocity of the particle, but there is no abrupt change in position. For since $\dot{\mathbf{r}}_0$ is a constant, we have, on integrating (59.1 a) with respect to t, the equation

$$m(\mathbf{r} - \mathbf{r}_0) - m\dot{\mathbf{r}}_0(t - t_0) = \int_{t_0}^{t} \mathbf{I}(t) dt.$$

As $t \rightarrow t_0$, this gives $\mathbf{r} - \mathbf{r}_0 \rightarrow 0$, since $\mathbf{I}(t)$ is supposed to remain finite as $t \rightarrow t_0$.

The general dynamic theorems having reference to impulsive forces acting at an instant $t = t_0$ on a system of material particles may be obtained from the corresponding theorems on continuous forces by integrating them with respect to the time

t from t_0 to t and then letting $t \to t_0$. The positions of the various particles of the system may be regarded as fixed in the time integration, and continuous forces may be neglected since, by the definition of impulse, the impulse of a continuous force at any instant $t = t_0$ is zero. The impulse moment about the origin of an impulsive force $\mathbf{F}_r$ is

$$\underset{t \to t_0}{\mathrm{Lt}} \int_{t_0}^{t} [\mathbf{r}.\mathbf{F}_r]dt \quad \text{or} \quad [\mathbf{r}.\mathbf{I}], \tag{59.2}$$

since $\mathbf{r}$ may be regarded as constant in the time integration. On integrating the theorem that the time derivative of the momentum system of localized vectors is equivalent to the external-applied-relative-force system, we find that, at any instant $t = t_0$, *the change in the momentum localized-vector system is equivalent to the external-relative-impulse localized-vector system.* Equating the resultant vector **R** of the two localized-vector systems, we see that the change in velocity of the center of mass is the same as if the total mass of the system were concentrated there and all the impulses acted through the center of mass. Equating the two moment vectors **G** of the localized-vector systems, we see that *the change in the angular momentum of the system about the origin is equal to the moment of all the applied impulses about the origin.*

If we have a rigid body turning about a fixed point, and we choose as reference frame the principal axes of inertia at the fixed point, the change in angular momentum of the body about the origin is the vector whose components are $A(\omega'_x - \omega_x)$, $B(\omega'_y - \omega_y)$, $C(\omega'_z - \omega_z)$, where $(\omega_x, \omega_y, \omega_z)$, $(\omega'_x, \omega'_y, \omega'_z)$ are the components of the rotation vector before and after the application of the impulse, respectively. If $(\Lambda, \mathrm{M}, \mathrm{N})$ are the components of the moment of the impulse, we have

$$A(\omega'_x - \omega_x) = \Lambda, \quad B(\omega'_y - \omega_y) = \mathrm{M}, \quad C(\omega'_z - \omega_z) = \mathrm{N}. \tag{59.3}$$

If the body is initially at rest, we have

$$A\omega'_x = \Lambda, \quad B\omega'_y = \mathrm{M}, \quad C\omega'_z = \mathrm{N}. \tag{59.3 a}$$

These equations show that the body will commence turning about that radius vector of the momental ellipsoid

$$Ax^2 + By^2 + Cz^2 = 1$$

which joins the fixed point to the point of contact with the ellipsoid of a tangent plane perpendicular to the impulse moment (Λ, M, N). As an example, let us consider the case of a body which at a given instant $t = t_0$ is turning about a fixed point O with angular velocity $\boldsymbol{\omega}$. At the instant $t = t_0$ a second point O' of the body is suddenly fixed. This implies the application of an impulsive force at O' and of an impulsive reaction at O. Hence the impulse moment about $\overrightarrow{OO'}$ is zero. Consequently the angular momentum of the body about OO' is the same before and after the fixture. If ω' is the magnitude of the angular velocity about $\overrightarrow{OO'}$ after the fixture, the angular momentum about $\overrightarrow{OO'}$ after the fixture is $(Al^2 + Bm^2 + Cn^2)\omega'$, where (l, m, n) are the direction cosines of OO' referred to the principal axes at O (see § 55). (The moment of inertia about $\overrightarrow{OO'}$ is $Al^2 + Bm^2 + Cn^2$.) The angular momentum about $\overrightarrow{OO'}$ just before the fixture is $A\omega_x l + B\omega_y m + C\omega_z n$, and so ω' is given by

$$\omega' = \frac{A\omega_x l + B\omega_y m + C\omega_z n}{Al^2 + Bm^2 + Cn^2}.$$

The body will be reduced suddenly to rest if OO' is perpendicular to the initial angular-momentum vector $(A\omega_x, B\omega_y, C\omega_z)$.

EXAMPLES

1. A number of uniform rods of masses $M_1, M_2, \cdots, M_n$ are smoothly jointed to each other in succession and laid in a straight line on a smooth table. If the end of the rod of mass M_n is free and the other end is suddenly moved with velocity V in a direction perpendicular to the line of the rods, then the initial velocities of the joints 12, 23, $\cdots$, and of the free end are $V_1, V_2, \cdots, V_n$, where

$$M_1(V + 2V_1) + M_2(2V_1 + V_2) = M_2(V_1 + 2V_2) + M_3(2V_2 + V_3)$$
$$= \cdots = V_{n-1} + 2V_n = 0.$$

Solution. Denoting the lengths of the rods by $(2l_1, 2l_2, \cdots, 2l_n)$, their initial angular velocities are $\frac{V_1 - V}{2l_1}, \frac{V_2 - V_1}{2l_2}, \cdots, \frac{V_n - V_{n-1}}{2l_n}$ respectively. The initial velocities of the centers of mass are $\frac{V + V_1}{2}$, $\frac{V_1 + V_2}{2}, \cdots, \frac{V_{n-1} + V_n}{2}$ respectively. Since the joints are smooth, the impulsive action and reaction at any joint are each perpendicular to the line of rods; and if we denote the magnitude of the impulse applied at the end of the first rod by R, and that of the impulse at the first joint by $\pm R_1$, and so on, we have

$$\frac{M_1(V+V_1)}{2} = R - R_1,$$

$$\frac{M_2(V_1+V_2)}{2} = R_1 - R_2,$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{M_n(V_{n-1}+V_n)}{2} = R_{n-1}.$$

Similarly, on taking moments about the center of mass of each rod, we obtain the equations

$$\frac{1}{3} M_1 l_1^2 \left(\frac{V_1 - V}{2\, l_1}\right) = -\,(R + R_1) l_1,$$

$$\frac{1}{3} M_2 l_2^2 \left(\frac{V_2 - V_1}{2\, l_2}\right) = -\,(R_1 + R_2) l_2,$$

etc.

We have, then, the two pairs of equations

$$R - R_1 = \frac{M_1}{2}(V + V_1), \qquad R + R_1 = \frac{M_1}{6}(V - V_1),$$

and $$R_1 - R_2 = \frac{M_2}{2}(V_1 + V_2), \quad R_1 + R_2 = \frac{M_2}{6}(V_1 - V_2),$$

from each of which R_1 may be obtained. Upon equating the values of R_1 found in this way, we have

$$M_1(V + 2\,V_1) + M_2(2\,V_1 + V_2) = 0.$$

The other equations are found in the same way. (The last equation is found immediately by equating the two expressions for R_{n-1}.)

2. A homogeneous rod of mass M and length $2\,a$ moves on a horizontal plane, one end being constrained to move without friction in a fixed straight line. The rod is initially perpendicular to the line and is struck at the free end by a blow I parallel to the line. Show that after a time t the perpendicular distance of the middle point of the rod from the line is given by the equation

$$\int_{\frac{y}{a}}^{1}\left(1 - \frac{3}{4}s^2\right)^{\frac{1}{2}}(1 - s^2)^{-\frac{1}{2}} ds = \frac{3\,It}{2\,Ma}.$$

Solution. The impulsive reaction of the fixed straight line is perpendicular to the line, since the latter is smooth. Hence the initial velocity of the center of mass is $\frac{I}{M}$ in a direction parallel to the line, and the initial angular velocity $\dot{\theta}_0$ is given by $\frac{1}{3}\,Ma^2\dot{\theta}_0 = -\,Ia$ or

$\dot{\theta}_0 = \frac{-3I}{Ma}\cdot$ The rod now begins to move subject merely to the reaction of the smooth straight line. The coördinates of the center of mass being $(x + a\cos\theta,\ a\sin\theta)$, we have

$$M\frac{d^2}{dt^2}(x + a\cos\theta) = 0, \quad M\frac{d^2}{dt^2}(a\sin\theta) = R.$$

On taking moments about the center of mass, $\frac{1}{3}Ma^2\dot{\theta} = -Ra\cos\theta$. From these equations we obtain $\dot{x} - a\sin\theta\,\dot{\theta} = \text{constant} = \frac{I}{M}$, its initial value. Upon elimination of R, we have $\frac{d^2}{dt^2}(\sin\theta) = \frac{-\dot{\theta}}{3\cos\theta}$ or $\cos^2\theta\,\ddot{\theta} - \sin\theta\cos\theta\,\dot{\theta}^2 = \frac{-\dot{\theta}}{3}\cdot$ If we multiply this equation by $\dot{\theta}$, it integrates to $\dot{\theta}^2\left(\cos^2\theta + \frac{1}{3}\right) = \text{constant} = \frac{3I^2}{M^2a^2}$, its initial value. (This result might also have been obtained directly from the energy integral $T = \text{constant}$.) Since $y = a\sin\theta$, we have $\dot{\theta} = \frac{a\dot{y}}{\left(1 - \frac{y^2}{a^2}\right)^{\frac{1}{2}}}$.

Combining this with $\dot{\theta}^2\left(\cos^2\theta + \frac{1}{3}\right) = \frac{3I^2}{M^2a^2}$, and writing s for $\sin\theta$, we obtain the result stated.

EXERCISES

1. A uniform rod at rest is struck at one end by an impulse at right angles to its length. Prove that if the rod is free, it begins to turn about a point which is one third of its length distant from the other end, and that the kinetic energy generated is greater, in the ratio 4 : 3, than it would be if the other end were fixed.

2. An elliptic disk is rotating in its plane about one end P of a diameter PP' when P' is suddenly fixed. Find the impulse at P and the angular velocity about P. Prove that if the eccentricity exceeds $(\frac{2}{3})^{\frac{1}{2}}$, the diameter PP' may be so chosen that the disk is reduced to rest.

3. A particle of mass m is attached by inextensible threads to particles of masses m' and m''. The particles are placed on a smooth table, with the threads in two perpendicular lines, and the particle m is struck by a blow in the direction of the bisector of the angle between the threads so that both threads are jerked. Prove that the initial velocities of m' and m'' are in the ratio $m + m'' : m + m'$.

4. Two equal rigid rods AB, BC, of negligible masses, carry equal particles attached at A, C, and the middle points of the rods. The rods being freely hinged at B and laid out straight, the end A is struck with an impulse at right angles to the rods. Prove that the velocities of the particles are in the ratios 9 : 2 : 2 : 1.

5. A target consists of a square plate of metal of edge a and mass M, hinged about its highest edge, which is horizontal. When at rest it is struck, at a point of depth h below the line of hinges, by an inelastic bullet of small mass m moving with velocity v. Find the subsequent motion of the target.

6. A rigid body is capable of turning freely about a fixed axis. Show that an impulse must act perpendicularly to the plane containing the axis of rotation and the center of mass of the body, and that the axis of rotation must be a principal axis of inertia at some point of its length, in order that there may be no impulsive pressure on the axis when the body is struck by a blow.

7. A circular lamina rests on a smooth horizontal table. Show that if, when struck, it begins to turn around a point on its circumference, the line of action of the blow divides the perpendicular diameter in the ratio 3 : 1.

8. A shot whose mass is m penetrates a thickness s of a fixed plate of mass M. Prove that if M is free to move, the thickness penetrated is $\dfrac{s}{1+\dfrac{m}{M}}$.

9. Four equal particles are connected by three equal strings AB, BC, CD, and lie on a horizontal plane with the strings taut in the form of half a regular hexagon. An impulse is applied at A in the direction DA. Prove that the initial tension of BC is one fourteenth of the impulse.

CHAPTER IX

THE MOTION OF A RIGID BODY ABOUT A FIXED POINT

60. Geometric discussion. Let us consider the motion of a rigid body one of whose points is fixed. The body will be said to be balanced when the external-relative-force system is equivalent to a single segment through the fixed point, so that the moment of this external-relative-force system about the fixed point is zero. The body is then sometimes said to be moving "under the action of no forces," since the external relative forces, being equivalent to a single segment through the point of support, will be neutralized or balanced by the reaction of the support. When we have a rigid body without a fixed point, we may apply our results to the problem of the motion of the body relative to its center of mass, provided the external-relative-force system is equivalent to a single segment through the center of mass. For we may reduce the center of mass to rest by taking a new reference frame, with its origin at the center of mass, but without any rotation relative to the original frame. This will bring in, on each particle of the system, an additional *centrifugal* force equal to the mass of the particle times the reversed acceleration of the center of mass relative to the original frame. However, this additional force system, which must be added to the original relative-force system to get the force system relative to the new framework of reference, is equivalent to a single segment through the center of mass, so that if the original relative-force system was equivalent to a single segment through the center of mass, the same will also be true of the new relative-force system.

Denoting the fixed point — or the center of mass, as the case may be — by O, and taking the principal axes of inertia at O as our axes of reference, we have the Eulerian equations (§ 58)

$$A\frac{dp}{dt}=(B-C)qr,\quad B\frac{dq}{dt}=(C-A)rp,\quad C\frac{dr}{dt}=(A-B)pq,\qquad(60.1)$$

where (p, q, r) are the components, in the moving reference frame formed by the principal axes of inertia at O, of the angular-velocity vector $\boldsymbol{\omega}$ of the body. Upon multiplying these equations by (p, q, r) respectively and adding, we obtain, on integration with respect to t, the energy integral

$$Ap^2 + Bq^2 + Cr^2 = 2\,T, \tag{60.2}$$

where T is the constant kinetic energy of the body (§ 55). Similarly, on multiplying the equations (60.1) by (Ap, Bq, Cr) respectively and adding, we obtain, on integration with respect to t, the angular-momentum integral

$$A^2p^2 + B^2q^2 + C^2r^2 = G^2, \tag{60.3}$$

where G is the magnitude of the constant angular-momentum vector $\mathbf{G}$ of the body about O.

The end point (p, q, r) of the angular-velocity vector $\boldsymbol{\omega}$ lies, accordingly, on the intersection of the two ellipsoids

$$Ax^2 + By^2 + Cz^2 = 2\,T, \tag{60.4}$$

$$A^2x^2 + B^2y^2 + C^2z^2 = G^2. \tag{60.5}$$

The ellipsoid (60.4) is similar in form and similarly situated to the momental ellipsoid, $Ax^2 + By^2 + Cz^2 = 1$, of the body at O, and is known as the Poinsot ellipsoid. By means of it Poinsot gave a convenient geometric description of the motion of the body. Since the tangent plane to (60.4) at the point (p, q, r) is $Apx + Bqy + Crz = 2\,T$, this tangent plane is perpendicular to the angular-momentum vector $\mathbf{G} = (Ap, Bq, Cr)$; and so its direction is fixed. (The direction of $\mathbf{G}$ is *from* O *to* the tangent plane $Apx + Bqy + Crz = 2\,T$; for the scalar product $(\boldsymbol{\omega}.\mathbf{G}) = Ap^2 + Bq^2 + Cr^2$ is positive, so that the angle between $\boldsymbol{\omega}$ and $\mathbf{G}$ is an acute one.) The distance of the tangent plane from O is the constant $\frac{2\,T}{G}$. Therefore the plane $Apx + Bqy + Crz = 2\,T$ is a fixed plane. Its point of contact (p, q, r) with the Poinsot ellipsoid, being on the axis of rotation, has zero velocity; and so we may describe the motion of the rigid body about O by the statement that a certain ellipsoid (the Poinsot ellipsoid), which is fixed in the body, rolls without slipping upon a fixed plane (known as the invariable plane). By the word "roll" no more is meant here than that the point of the rigid body (or of the attached ellipsoid) which is in contact with the invariable plane has zero velocity. Usually the rolling of one surface upon another implies that the axis of rotation passes through the point of contact and lies in the tangent plane common to the two surfaces. The word "pivoting" is used when the axis of rotation lies along the common normal to the two surfaces. In the present case the rotation vector $\boldsymbol{\omega}$ is, in general, oblique to the invariable plane, so that the term "rolling" implies a combination of pure rolling and pivoting.

The essential content of Poinsot's description is that the motion of the rigid body depends directly on the form of the momental ellipsoid at the fixed point O and only indirectly on the form of the rigid body itself. The end point of the rotation vector $\boldsymbol{\omega}$, when viewed from the moving body, traces out part of the curve of intersection of the two ellipsoids (60.4) and (60.5). This curve is known as the polhode curve, and the cone formed by joining its points to the fixed point O is known as the polhode cone. When viewed from our fixed reference frame, the end point of the rotation vector $\boldsymbol{\omega}$ traces out a plane curve lying in the invariable plane. This curve is called the herpolhode curve, and the cone formed by joining its points to O is called the herpolhode cone. We may therefore describe the motion of the body as the rolling of a cone which is rigidly attached to the body (the polhode cone) upon a cone which is fixed in space (the herpolhode cone), both cones having the same vertex O.

The equation of the polhode cone is found by multiplying (60.4) by G^2 and (60.5) by $2T$ and then subtracting the resulting equations. It is

$$Ax^2(G^2 - 2AT) + By^2(G^2 - 2BT) + Cz^2(G^2 - 2CT) = 0, \quad (60.6)$$

so that $\dfrac{G^2}{2T}$ is equal to or less than the greatest and equal to or greater than the least of the three principal moments of inertia (A, B, C) at O. The projections of the polhode curve on the three coördinate planes are found from (60.4) and (60.5) by eliminating, in turn, x^2, y^2, and z^2. They are

$$\begin{aligned} B(A-B)y^2 + C(A-C)z^2 &= 2AT - G^2, \\ C(B-C)z^2 + A(B-A)x^2 &= 2BT - G^2, \\ A(C-A)x^2 + B(C-B)y^2 &= 2CT - G^2. \end{aligned} \quad (60.7)$$

Let us arrange the axes so that B is the mean moment of inertia, that is, either $A \geqq B \geqq C$ or $A \leqq B \leqq C$. Then the projections of the polhode curve on the (y, z) and (x, y) coördinate planes are ellipses, and the projection on the (z, x) coördinate plane is a hyperbola. The axes will be named so that the x-axis is the transverse axis of the hyperbola; that is, we have the x-axis as the principal axis of greatest or least moment according as G^2 is greater or less than $2\,BT$. (When $G^2 = 2\,BT$, the hyperbola degenerates into a pair of straight lines $C(B - C)z^2 = A(A - B)x^2$. This case will be discussed in detail separately.) According to this convention as to the naming of the axes, the polhode curve, which is always closed (since its projections on the (y, z) and (x, y) coördinate planes are closed), surrounds the x-axis. When G^2 is very nearly equal to $2\,AT$ or $2\,CT$, the polhode curve does not depart very far from the x-axis. In these cases the axis of rotation is said to be stable, so that the principal axes of greatest and least moment of inertia at O are stable axes of rotation. The principal axis of mean moment of inertia is an unstable axis of rotation, since, when G^2 is nearly equal to $2\,BT$, the polhode curve is not very different from one of the ellipses obtained by intersecting the Poinsot ellipsoid by the planes

$$C(B - C)z^2 = A(A - B)x^2,$$

both of which pass through the y-axis. The eccentricity of the ellipse

$$B(A - B)y^2 + C(A - C)z^2 = 2\,AT - G^2$$

may be used to give an indication of the degree of stability of the principal axis of inertia of moment A. The smaller the eccentricity the greater the stability. Similarly, the eccentricity of the ellipse

$$A(A - C)x^2 + B(B - C)y^2 = G^2 - 2\,CT$$

measures the degree of stability of the principal axis of inertia of moment C.

61. Analytic discussion of the polhode curve. We shall suppose at first that no two of the three principal moments of inertia (A, B, C) at O are equal, and shall discuss afterwards the simpler case where two of these principal moments are equal. According to the convention already adopted as to the naming of the axes, the x-axis is that one which is surrounded by the polhode curve, and the y-axis is the principal axis of mean moment of inertia. This means that $A > B > C$ or $A < B < C$

according as $G^2 >$ or $< 2\,BT$ (the case $G^2 = 2\,BT$ is reserved for a special discussion). On solving the three equations

$$\begin{aligned} Ap^2 + Bq^2 + Cr^2 &= 2\,T, \\ A^2p^2 + B^2q^2 + C^2r^2 &= G^2, \\ p^2 + q^2 + r^2 &= \omega^2, \end{aligned}$$

the three quantities (p^2, q^2, r^2) may be expressed in terms of ω^2, the square of the magnitude of the rotation vector. The results are

$$\begin{aligned} p^2 &= \frac{BC(\omega^2 - \lambda_1)}{(A-B)(A-C)}, \\ q^2 &= \frac{CA(\omega^2 - \lambda_2)}{(B-C)(B-A)}, \\ r^2 &= \frac{AB(\omega^2 - \lambda_3)}{(C-A)(C-B)}, \end{aligned} \tag{61.1}$$

where

$$\begin{aligned} \lambda_1 &= \frac{2\,T(B+C) - G^2}{BC}, \\ \lambda_2 &= \frac{2\,T(C+A) - G^2}{CA}, \\ \lambda_3 &= \frac{2\,T(A+B) - G^2}{AB}. \end{aligned}$$

The greatest value that G^2 can have is $2\,AT$ if A is the greatest of the three principal moments of inertia at O; and so λ_1 is positive, since $B + C > A$. Moreover,

$$\lambda_3 - \lambda_1 = \frac{(G^2 - 2\,BT)(A-C)}{ABC}$$

and

$$\lambda_2 - \lambda_3 = \frac{(2\,AT - G^2)(B-C)}{ABC},$$

so that $\lambda_1 < \lambda_3 < \lambda_2$. According to our convention as to the choice of axes, this last pair of inequalities holds also when A is the least, instead of the greatest, of the three principal moments of inertia at O. It follows, from the expressions (61.1), that ω^2 must lie in value between λ_3 and λ_2. For $\omega^2 - \lambda_2$ cannot be positive, since $(B-C)(B-A)$ is negative; and $\omega^2 - \lambda_3$ cannot be negative, since $(C-A)(C-B)$ is positive. Hence $\lambda_3 \leqq \omega^2 \leqq \lambda_2$, which says that the herpolhode curve must lie entirely in the ring between two concentric circles in the invariable plane, these circles having a common center at the foot of the perpendicular from O to this plane and radii whose squares are $\left(\lambda_3 - \frac{4T^2}{G^2}\right)$ and $\left(\lambda_2 - \frac{4T^2}{G^2}\right)$ respectively; for the distance from O to the invariable plane is $\frac{2T}{G}$.

Upon solving for p^2 and r^2 in terms of q^2 from the two equations (60.2) and (60.3), we obtain

$$\begin{aligned} p^2 &= \frac{[G^2 - 2CT - B(B-C)q^2]}{A(A-C)}, \\ r^2 &= \frac{[2AT - G^2 - B(A-B)q^2]}{C(A-C)}, \end{aligned} \tag{61.2}$$

and, on substituting these expressions in the second of the Eulerian equations (60.1), we have

$$\begin{aligned} B\frac{dq}{dt} &= \left[\frac{(G^2 - 2CT)(2AT - G^2)}{AC}\right]^{\frac{1}{2}} \times \left[1 - \frac{B(B-C)q^2}{G^2 - 2CT}\right]^{\frac{1}{2}} \\ &\quad \times \left[1 - \frac{B(A-B)q^2}{2AT - G^2}\right]^{\frac{1}{2}}. \end{aligned} \tag{61.3}$$

Of the two coefficients of q^2 in the expressions under the radical on the right, $\frac{B(A-B)}{2AT - G^2}$ is the greater, as is readily seen upon taking their difference and remembering the convention as to the naming of the principal axes of inertia at O. Introducing, for the moment, the new variable q' defined by the equation

$$q = \beta q', \quad \beta = \left[\frac{2AT - G^2}{B(A-B)}\right]^{\frac{1}{2}}, \tag{61.4}$$

our equation (61.3) takes the form

$$\frac{dq'}{dt} = \sigma[(1 - q'^2)(1 - k^2q'^2)]^{\frac{1}{2}}, \tag{61.5}$$

where

$$\begin{aligned} k^2 &= \frac{(B-C)(2AT - G^2)}{(A-B)(G^2 - 2CT)} < 1, \\ \text{and} \quad \sigma &= \left[\frac{(G^2 - 2CT)(A-B)}{ABC}\right]^{\frac{1}{2}}. \end{aligned} \tag{61.6}$$

Hence

$$\sigma t = \int_0^{q'} \frac{ds}{[(1-s^2)(1-k^2s^2)]^{\frac{1}{2}}},$$

or, on making the change of variable $s = \sin\theta$,

$$\sigma t = \int_0^{\phi} \frac{d\theta}{[1 - k^2\sin^2\theta]^{\frac{1}{2}}} = F(\phi, k), \quad (\phi = \text{arc}\sin q') \tag{61.7}$$

(see § 45). We have so chosen the constant of integration that t is to be measured from $q' = 0$ or $q = 0$, that is, from an instant at which the angular-velocity vector $\boldsymbol{\omega}$ is in the (z, x) principal plane of inertia at O. Upon inverting the elliptic integral (61.7), we have

$$q' = \operatorname{sn} \sigma t,$$

so that

$$q = \beta \operatorname{sn} \sigma t.$$

Upon substituting this value in the expressions (61.2), and using the relations $\operatorname{cn}^2 v = 1 - \operatorname{sn}^2 v$, $\operatorname{dn}^2 v = 1 - k^2 \operatorname{sn}^2 v$, of § 46, we obtain

$$p = \alpha \operatorname{dn} \sigma t, \qquad r = \gamma \operatorname{cn} \sigma t,$$

where

$$\alpha = \left[\frac{G^2 - 2\,CT}{A(A - C)}\right]^{\frac{1}{2}}, \quad \gamma = \left[\frac{2\,AT - G^2}{C(A - C)}\right]^{\frac{1}{2}}. \tag{61.8}$$

The values of α and γ may be conveniently remembered by observing that they are the initial values, respectively, of p and r and that the initial value of q is zero, so that (60.2) and (60.3) give

$$A\alpha^2 + C\gamma^2 = 2\,T, \quad A^2\alpha^2 + C^2\gamma^2 = G^2.$$

The values of β and σ then follow on substituting the values

$$p = \alpha \operatorname{dn} \sigma t, \quad q = \beta \operatorname{sn} \sigma t, \quad r = \gamma \operatorname{cn} \sigma t \tag{61.9}$$

directly in the Eulerian equations (60.1). This substitution serves also to remove the ambiguities in the signs of α, β, γ, and σ when these are defined by the equations (61.4), (61.6), and (61.8). We observe, first of all, that there is no lack of generality in assuming σ positive; for a change in the sign of σ, coupled with a change in the sign of β, the signs of α and γ remaining unaltered, leads to the same values of (p, q, r). (The functions cn and dn are even, and sn is an odd function.) We derive the three equations

$$\begin{aligned} -k^2\sigma A\alpha &= (B - C)\beta\gamma, \\ \sigma B\beta &= (C - A)\gamma\alpha, \\ -\sigma C\gamma &= (A - B)\alpha\beta, \end{aligned} \tag{61.10}$$

from which we can solve for β, σ, and k^2 in terms of α and γ. Once the signs of α and γ have been fixed (by giving the arbitrary initial angular-velocity vector $(\alpha, 0, \gamma)$), the sign of β is determined by the fact that $\dfrac{\alpha\beta}{\gamma}$ must have the same sign as $B - A$.

62. The motion of the body in space. The values of (p, q, r) given in the preceding paragraph furnish the parametric equations of the polhode curve; that is, they describe the motion of the axis of rotation in the body. In order to describe the motion of the body relative to a fixed reference frame $O\xi\eta\zeta$, we avail ourselves of the Eulerian angles (θ, ϕ, ψ) of § 23. Taking the constant direction of the angular-momentum vector about O as the direction of the $O\zeta$ axis of the fixed frame, the direction cosines of $O\zeta$ relative to the moving frame $OXYZ$ are found, by the table (26.8), to be $(-\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$. Hence we have the three equations

$$\begin{aligned} Ap &= -G\sin\theta\cos\psi, \\ Bq &= G\sin\theta\sin\psi, \\ Cr &= G\cos\theta. \end{aligned} \tag{62.1}$$

From the last of these equations we obtain

$$\cos\theta = C\gamma\frac{\operatorname{cn}\sigma t}{G}. \tag{62.2}$$

On eliminating θ from the first two equations (62.1), we derive

$$\tan\psi = \frac{-Bq}{Ap} = \frac{-B\beta\operatorname{sn}\sigma t}{A\alpha\operatorname{dn}\sigma t}, \tag{62.3}$$

the quadrant in which ψ lies being determined by the fact that $\sin\psi$ has the same sign as q. The determination of the remaining angle ϕ is not so direct. Upon writing p and q for ω_x and ω_y, respectively, in (26.10), we find

$$\dot\phi\sin\theta = q\sin\psi - p\cos\psi,$$

or, by (62.1),

$$\dot\phi = \frac{(Bq^2 + Ap^2)}{G\sin^2\theta} = \frac{G(2T - Cr^2)}{G^2 - G^2\cos^2\theta} = \frac{G(2T - Cr^2)}{G^2 - C^2r^2}.$$

Hence

$$\phi = G\int_0^t \frac{(2T - Cr^2)}{G^2 - C^2r^2}\,dt,$$

where

$$r = \gamma\operatorname{cn}\sigma t. \tag{62.4}$$

The constant of integration has been chosen so as to make $\phi = 0$ when $t = 0$, that is, when the axis of rotation is in the (z, x) plane of the body. Hence the $O\eta$ axis in space coincides initially with the line of nodes.

63. Discussion of the special case $G^2 = 2BT$. In this case the polhode curve consists of two ellipses, and the previous formulas degenerate into formulas not involving the elliptic functions. In the solution already given for the general case the expression k^2 of (61.6) was less than unity; but when $G^2 = 2\,BT$, k^2 becomes unity, so that the elliptic integral of (61.7) becomes

$$\sigma t = \int_0^{q'} \frac{ds}{1 - s^2},$$

σ having the same expression as before. Hence $q' = \tanh \sigma t$, giving

$$q = \beta \tanh \sigma t, \quad \beta = \left(\frac{2\,T}{B}\right)^{\frac{1}{2}} = \pm \frac{G}{B}. \tag{63.1}$$

Proceeding as before, we find

$$p = \alpha \operatorname{sech} \sigma t, \qquad r = \gamma \operatorname{sech} \sigma t,$$

where
$$\alpha = \left[\frac{2\,T(B - C)}{A(A - C)}\right]^{\frac{1}{2}}, \quad \gamma = \left[\frac{2\,T(A - B)}{C(A - C)}\right]^{\frac{1}{2}}. \tag{63.2}$$

Without any lack of generality, σ may be assumed positive, and the signs to be attributed to α and γ may be either positive or negative (they are determined by the initial conditions), the sign to be attributed to β being then determined by the fact that $\frac{\alpha\beta}{\gamma}$ must have the same sign as $B - A$. Naming the axes so that $A > B > C$, we see that $\frac{\alpha}{\gamma}$ has the opposite sign to β. But $\frac{dq}{dt} = \beta\sigma \operatorname{sech}^2 \sigma t$, so that it has the same sign as β and therefore the opposite sign to $\frac{\alpha}{\gamma}$. As t increases indefinitely, both p and r tend to zero, and q tends to the value β. Hence the axis of rotation tends steadily to coincidence with the principal axis of mean moment of inertia at O. Its direction is the same as, or opposite to, that of this principal axis, according as the initial values of p and r have the same sign or not.

To describe the motion of the body in space we have the equations

$$\cos \theta = \frac{C\gamma \operatorname{sech} \sigma t}{G},$$
$$\tan \psi = -\frac{Bq}{Ap} = -\frac{B\beta \sinh \sigma t}{A\alpha}, \tag{63.3}$$
$$\phi = G\int_0^t \frac{2\,T - Cr^2}{G^2 - C^2 r^2}\,dt. \quad (r = \gamma \operatorname{sech} \sigma t)$$

As t increases indefinitely, θ tends to $\frac{\pi}{2}$, and ψ tends to $\pm \frac{\pi}{2}$, the positive sign being used when q is positive for large values of t (see (62.1)). This means that β is positive or that p and r have initially opposite signs. Moreover, $\dot{\phi}$ tends to $\frac{2T}{G}\cdot$ From the table (26.8) we see that the axis of mean moment tends to set itself along the angular-momentum-about-O vector, having the same or opposite direction according as p and r have initially opposite signs or not. The body tends asymptotically to a state of rotation with constant angular velocity $\frac{2T}{G}$ around the fixed angular-momentum line.

64. The dynamically symmetrical body. We have supposed heretofore that the three principal moments of inertia at O were unequal. When two or more of these moments are equal, the problem of integrating the Eulerian equations (60.1) is considerably simplified. First of all, we may deal with the case where the fixed point O is a spherical point, that is, where $A = B = C$. We find at once, from (60.1), that (p, q, r) are all constant. Hence the axis of rotation is fixed in the body, and it follows that it is fixed also in space. The absolute velocity of any point is $\frac{d\mathbf{r}}{dt} + [\boldsymbol{\omega}.\mathbf{r}]$. This is zero for points rigidly attached to the body ($\mathbf{r}$ constant) and lying on the axis of rotation ($[\boldsymbol{\omega}.\mathbf{r}] = \mathbf{0}$). Hence the motion of the body is one of rotation with constant angular velocity about a permanent axis, that is, an axis which is fixed in the body and in space.

A more interesting case is that where two of the principal moments of inertia at O are equal without all three being equal. The axis of unequal moment is called simply the *axis of figure* of the body, and any axis through O perpendicular to the axis of figure is a principal axis of inertia at O. We shall suppose the axes named so that $A = B \neq C$. The axis of figure is, accordingly, the z-axis of a reference frame which moves with the body. The third Eulerian equation (60.1) gives $\frac{dr}{dt} = 0$, so that r is a constant. Denoting its constant value by n, we have, from the second of the equations (61.2),

$$n = \left[\frac{2AT - G^2}{C(A - C)}\right]^{\frac{1}{2}}, \tag{64.1}$$

the sign being determined by the initial conditions. The first two Eulerian equations now become

$$A\frac{dp}{dt} = (A - C)qn, \quad A\frac{dq}{dt} = (C - A)pn.$$

Upon multiplying the second of these by the imaginary unit i and adding it to the first, we obtain

$$A\frac{d}{dt}(p+iq) = -i(A-C)n(p+iq),$$

of which the integral is

$$p+iq = \beta e^{\frac{i(C-A)nt}{A}},$$

β being real if q is initially equal to zero. Supposing this to be the case, we have

$$p = \beta\cos\sigma t, \quad q = \beta\sin\sigma t, \quad \sigma = (C-A)\frac{n}{A}. \qquad (64.2)$$

The value of β is determined by the fact that, since our energy and angular-momentum integrals now take the forms

$$A(p^2+q^2)+Cn^2 = 2\,T, \quad A^2(p^2+q^2)+C^2n^2 = G^2,$$

$$p^2+q^2 = \frac{G^2-2\,CT}{A(A-C)};$$

whence

$$\beta = \left[\frac{G^2-2\,CT}{A(A-C)}\right]^{\frac{1}{2}}, \qquad (64.3)$$

the sign to be attributed to β being the same as that of the initial value of p.

The results $p = \beta\cos\sigma t, \quad q = \beta\sin\sigma t, \quad r = n$

show that the polhode cone is a right circular cone, its axis being the axis of figure, and the half-angle δ at the vertex having for its tangent the numeric magnitude of

$$\frac{\beta}{n}. \qquad (64.4)$$

The magnitude of the angular-velocity vector ω is

$$\omega = (p^2+q^2+r^2)^{\frac{1}{2}} = (\beta^2+n^2)^{\frac{1}{2}} = \left[\frac{2\,T(A+C)-G^2}{AC}\right]^{\frac{1}{2}}, \qquad (64.5)$$

so that the herpolhode cone is also a right circular cone, its axis being the angular-momentum or invariable line. The semiangle ϵ at its vertex is given by

$$\cos\epsilon = \frac{2\,T}{G\omega}. \qquad (64.6)$$

In order to describe the motion of the body in space, we use the same Eulerian angles (θ, ϕ, ψ) as in the general case. From the relation

$$\cos\theta = \frac{Cn}{G}, \tag{64.7}$$

we see that the axis of figure describes a right circular cone about the invariable line. From

$$\tan\psi = \frac{-Aq}{Ap} = -\tan\sigma t$$

we obtain

$$\psi = -\sigma t \quad \text{or} \quad \pi - \sigma t. \tag{64.8}$$

Which of these two expressions to take is determined by the fact that since $Bq = G\sin\theta\sin\psi$, $\sin\psi$ has the same sign as q. Hence, if β is positive, $\psi = \pi - \sigma t$; whereas if β is negative, $\psi = -\sigma t$. The angular velocity $\dot{\psi}$ of the body round its axis of figure is constant and equal to $-\sigma$; that is,

$$\dot{\psi} = (A - C)\frac{n}{A}. \tag{64.9}$$

This is known as the spin of the body. To find the angular velocity $\dot{\phi}$ round the invariable line, we make use of the third of the equations (26.10), $r = \dot{\psi} + \dot{\phi}\cos\theta$ or, here, $n = -\sigma + \dot{\phi}\frac{Cn}{G}$. From this we have

$$\dot{\phi} = \frac{G}{A}. \tag{64.10}$$

Since the angular velocity $\dot{\theta}$ around the line of nodes is here zero, the angular-velocity vector $\boldsymbol{\omega}$ must lie in the plane of the invariable line and the axis of figure. This plane is known as the plane of precession, and it moves around the invariable line with constant angular velocity $\dot{\phi} = \frac{G}{A}$. On combining (64.9) with $n = \dot{\psi} + \dot{\phi}\cos\theta$, we obtain the useful relation

$$C\dot{\psi} = (A - C)\dot{\phi}\cos\theta, \tag{64.11}$$

connecting the spin $\dot{\psi}$, the angular velocity of precession $\dot{\phi}$, and the angle of precession θ.

The particular direction along the axis of figure which we agree to choose as that of the z-axis is at our disposal, and we shall make the choice which gives a positive spin $\dot{\psi}$. This requires that n have the sign of $A - C$ (see (64.9)). Since $\dot{\psi}$ and $\dot{\phi}$ are both positive, *the axis of rotation must lie between the* *axis of figure and the invariable line.* There are two distinct cases, which are distinguished from one another by the sign of n or, equivalently, by the relative magnitude of A and C.

CASE I. *The direct, or epicycloidal, precession.* Here $A > C$, and n is positive. Since n measures the projection of ω upon the axis of figure, the angle between the axis of figure and ω is an acute one; it is the semivertical angle δ of the polhode cone. Moreover, the angle θ is acute, since $\cos\theta = \frac{Cn}{G}$ is positive. The semivertical angle ϵ of the herpolhode cone is always acute, since the scalar product $(\mathbf{G}.\omega) = 2\,T$ is positive. So we may describe the motion of the dynamically symmetrical body for which $A > C$, C being the unequal moment, by saying that the polhode cone rolls *externally* on the herpolhode cone. (See Fig. 29.)

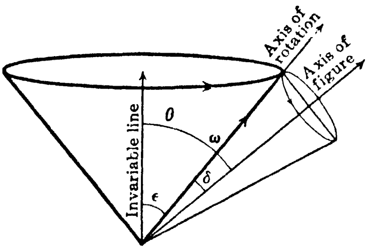

FIG. 29

The relation between the angle of precession θ and the semivertical angles δ and ϵ of the polhode and herpolhode cones, respectively, is

$$\theta = \delta + \epsilon. \qquad (64.12)$$

This type of motion is known as the direct, or epicycloidal, precession, the term "epicycloidal" coming from the analogy of this motion with the generation of epicycloids by the rolling of one circle *externally* on another.

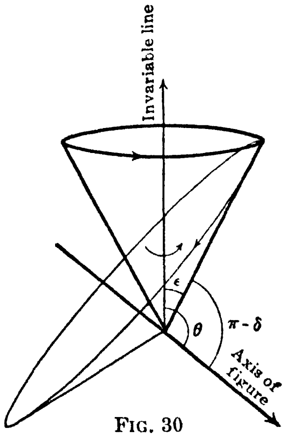

FIG. 30

CASE II. *The retrograde, or pericycloidal, precession.* Here $A < C$, and n is negative. The angle $\pi - \delta$ between the axis of figure and the axis of rotation is obtuse, as is also the angle θ. The polhode cone envelops the herpolhode cone on which it rolls externally. (See Fig. 30.) The relation between θ and the openings δ and ϵ of the two cones is

$$\theta = \epsilon + \pi - \delta. \qquad (64.13)$$

Since the axis of rotation always lies between the invariable line and the axis of figure, the polhode cone cannot roll *internally* on the herpolhode cone; that is, we cannot have a hypocycloidal precession.

65. Application to the rotation of the earth. If we regard the earth as built up of homogeneous spheroidal shells, it is dynamically symmetrical, since its polar axis is the axis of unequal moment. From the expression for the moment of inertia of an ellipsoid about one of its axes (Exercise 4, p. 197), we have

$$\frac{C}{A} = \frac{2\,a^2}{(a^2 + c^2)} = \frac{2}{1 + \dfrac{c^2}{a^2}},$$

where c is the axial and a the equatorial radius of the earth. As to the numeric magnitudes of these quantities, we have $\frac{c^2}{a^2} = .9933$ (approximately), so that

$$\frac{C}{A} = \frac{2}{1.9933}. \tag{65.1}$$

If, now, we regard the earth as rigid and assume that the gravitational force of the other heavenly bodies acting upon it is equivalent to a single force through the center of the earth, we may apply the theory just given (the center of the earth playing the rôle of the fixed point). First of all, we know that if the earth were once rotating around its polar axis, it would continue to rotate around this axis, which would maintain a fixed direction in space, with constant angular velocity. For the polar axis, being a principal axis of inertia at the center of mass, is a permanent axis of rotation (see § 58). The time taken by the earth to make one rotation with respect to a framework of reference attached to the fixed stars is known as the sidereal day and contains 86,164 seconds. Hence ω, when expressed in radians per second, is a very small number,

$$\omega = \frac{2\,\pi}{86{,}164} = .0000729. \tag{65.2}$$

For ordinary mechanical purposes it is sufficient to assume that this is the actual state of affairs. But astronomical observations show that the axis of the earth is not fixed in the earth, although its deviation from the polar axis is very slight. We shall therefore investigate the nature of the precessional motion of the earth, on the assumption that the axis of rotation is slightly different from the polar axis, the external forces being again supposed to have zero moment about the earth's center. Since $C > A$, the precession is of the retrograde type. For a first approximation $\theta = \pi$, and we have, from (64.11),

$$C\dot{\psi} = (C - A)\dot{\phi}. \tag{65.3}$$

Furthermore the vectors of magnitude $\dot{\psi}$ along the axis of figure and $\dot{\phi}$ along the invariable line have ω as their sum, so that

$$\frac{\dot{\phi}}{\sin \delta} = \frac{\dot{\psi}}{\sin \epsilon}. \tag{65.4}$$

Hence, δ and ϵ being each so small that the arc may be identified with the sine,

$$\frac{\delta}{\epsilon} = \frac{\dot{\phi}}{\dot{\psi}} = \frac{C}{C - A} = 300, \text{ approximately.}$$

The opening of the herpolhode cone is therefore only about one three-hundredth of the opening of the enveloping polhode cone.

The length of the sidereal day is $\frac{-2\pi}{n}$, where the negative number n is the resolved part of ω along the axis of figure of the earth (here supposed to be directed *from* the north pole *to* the south pole in order that $\dot{\psi}$ may be positive). Hence, from (64.9), the period of the rotation of the axis of rotation around the axis of figure is

$$\frac{2\pi}{\dot{\psi}} = \frac{2\pi A}{n(A - C)} = \frac{A}{C - A} = 300 \text{ sidereal days.}$$

The period of the precession is $\frac{2\pi}{\dot{\phi}} = \frac{2\pi(C - A)}{C\dot{\psi}} = .997$ of a sidereal day.

66. Motion relative to the earth. Neglecting the small changes in the axis of rotation of the earth, we shall assume that the earth is rotating with constant angular velocity $\frac{2\pi}{86{,}164}$ about its polar axis, which maintains a fixed direction in space. If we choose as our reference frame one fixed in the earth, with its origin at the center of the earth, this frame will be non-inertial, for two reasons:

1. The center of mass, owing to the gravitational influence of the sun etc., has an acceleration relative to an inertial frame.

We have seen, however, in § 51, that this acceleration is less than .02 feet per second per second. We shall neglect this, and shall suppose that the center of mass of the earth is at rest relative to an inertial frame.

2. The earth has an angular velocity ω about its polar axis relative to an inertial frame.

In order to obtain the relative force, $\mathbf{F}_r$, acting upon a particle *M which is at rest relative to the earth,* it is therefore necessary to add to the absolute forces acting on the particle a force $m\omega^2\overrightarrow{QM}$, where Q is the foot of the perpendicular from M to the polar axis of the earth (see (35.7)). Upon introducing space polar coördinates (r, θ, ϕ), with origin at the center of the earth, which we shall here regard as a sphere of radius a, and with polar axis along the join of the center of the earth to the north pole, the centrifugal force $m\omega^2\overrightarrow{QM}$ has, as components along our Cartesian axes of reference,

$$(ma\omega^2 \sin\theta\cos\phi,\ ma\omega^2\sin\theta\sin\phi,\ 0). \tag{66.1}$$

If we assume that the gravitational attraction of the earth on the particle is directed toward the center of the earth and is of magnitude mG, the components of the relative force on the particle are

$$\begin{gathered}(ma\omega^2\sin\theta\cos\phi - mG\sin\theta\cos\phi,\\ ma\omega^2\sin\theta\sin\phi - mG\sin\theta\sin\phi,\quad -mG\cos\theta).\end{gathered}$$

This vector has the magnitude mg, where

$$\begin{aligned} g &= [(a\omega^2 - G)^2\sin^2\theta + G^2\cos^2\theta]^{\frac{1}{2}}\\ &= G\left[1 - 2\frac{a\omega^2}{G}\sin^2\theta + \frac{a^2\omega^4\sin^2\theta}{G^2}\right]^{\frac{1}{2}}. \end{aligned} \tag{66.2}$$

The quantity g is known as the "acceleration due to gravity." The direction of this acceleration is known as the downward vertical. At the north or south pole, at either of which $\sin\theta = 0$, the value of g is G. Upon using $a = 4000 \times 5280$ feet (approximately), $a\omega^2$ turns out to be about $\frac{1}{9}$; and since G is about 32, the quantity $\frac{a\omega^2}{G}$ is about $\frac{1}{290}$, so that we may neglect its square in (66.2). We derive, then, the simpler formula

$$g = G\left(1 - \frac{a\omega^2}{G}\sin^2\theta\right). \tag{66.3}$$

The value 32 for G is obtained from the empirical formula

$$g = G\left(1 - \frac{\sin^2\theta}{191}\right).$$

The discrepancy between this and the formula (66.3),

$$g = G\left(1 - \frac{\sin^2\theta}{290}\right), \text{ approximately,}$$

can be accounted for by the fact that the earth is spheroidal and not spherical in shape.

It is essential to notice that when we speak of the "force of gravity" at the surface of the earth, we mean the sum of the absolute gravitational attraction of the earth and that part of the centrifugal force which is due to the drag acceleration relative to an inertial frame. When the particle under discussion has a velocity relative to the earth, there will be an additional term in the centrifugal force, namely, the centrifugal force due to the Coriolis acceleration. The effects of this we shall now proceed to discuss.

a. Free motion relative to the earth. We shall consider a particle at the surface of the earth, moving relatively to the earth but not departing far enough from its original position on the surface to oblige us to take into account the differences in g owing to the varying distance from the center of the earth and the varying latitude. Taking a reference frame fixed in the earth, with its origin at a point on the surface in the neighborhood of which the particle is moving, we obtain the relative force on the particle by adding to the force of gravity, $m\mathbf{g}$, the Coriolis part of the centrifugal force, namely, $2\,m[\mathbf{v}_r.\boldsymbol{\omega}]$. Let us take the upward vertical as our z-axis, the east line as our x-axis, and the north line as our y-axis. Then $\boldsymbol{\omega}$ has the components $(0, \omega\sin\theta, \omega\cos\theta)$, so that the components of the Coriolis part of the centrifugal force are

$$[2\,m\omega(\dot{y}\cos\theta - \dot{z}\sin\theta),\ -2\,m\omega\dot{x}\cos\theta,\ 2\,m\omega\dot{x}\sin\theta]. \qquad (66.4)$$

The equations of motion of the particle are

$$\begin{aligned} \ddot{x} &= 2\,\omega(\dot{y}\cos\theta - \dot{z}\sin\theta), \\ \ddot{y} &= -2\,\omega\dot{x}\cos\theta, \\ \ddot{z} &= 2\,\omega\dot{x}\sin\theta - g, \end{aligned} \qquad (66.5)$$

where θ is the colatitude of the point of the earth's surface at which the motion is taking place. Let us consider the case of a particle dropped from a height h with zero initial velocity. Then we have the following sets of initial values:

$$(x, y, z) = (0, 0, h), \quad (\dot{x}, \dot{y}, \dot{z}) = (0, 0, 0), \quad (\ddot{x}, \ddot{y}, \ddot{z}) = (0, 0, -g),$$

the last set following from (66.5). Upon differentiating (66.5) with respect to t, we find, for the initial values of the third derivatives, $(\dddot{x}, \dddot{y}, \dddot{z}) = (2\,\omega g \sin\theta, 0, 0)$; so that on expanding (x, y, z) by means of the Taylor series expansion

$$x = (x)_0 + (\dot{x})_0 t + (\ddot{x})_0 \frac{t^2}{2} + \cdots,$$

we find $$x = \frac{\omega g \sin\theta\, t^3}{3}, \quad y = 0, \quad z = h - \frac{gt^2}{2}. \tag{66.6}$$

In these expressions we have neglected terms involving powers of t above the third. The time of fall is found by putting $z = 0$ and is independent of ω (save in so far as g is affected by ω). There is, however, an easterly deviation from the vertical of amount $\frac{\omega g \sin\theta\, t^3}{3}$.

In general, the effect of the Coriolis part of the centrifugal force due to the rotation of the earth may be stated qualitatively from a consideration of the term $2\,m[\mathbf{v}_r . \boldsymbol{\omega}]$. If the relative velocity $\mathbf{v}_r$ of the particle is in a horizontal plane, the Coriolis force gives a relative acceleration whose components are $(2\,\omega \cos\theta\, \dot{y}, -2\,\omega\cos\theta\,\dot{x}, 2\,\omega\sin\theta\,\dot{x})$, the x and y axes being east and north respectively, and the z-axis being vertically upwards as before. Thus, for example, if the particle is moving north, there is an apparent deviating force toward the east in the Northern Hemisphere $\left(\theta < \frac{\pi}{2}\right)$ and toward the west in the Southern Hemisphere $\left(\frac{\pi}{2} < \theta < \pi\right)$. If the particle is moving east, there is an apparent deviating force toward the south in the Northern Hemisphere and toward the north in the Southern Hemisphere. In addition, there is an upward deviating force in both hemispheres (which becomes a downward deviating force when the particle is moving due west). This Coriolis centrifugal force seems to explain, at least qualitatively, the whirling motion of cyclones. Here we have a rush of particles of air in toward a point. The horizontal part of the Coriolis force $2\,m[\mathbf{v}_r . \boldsymbol{\omega}]$ is perpendicular to the radius vector to the point from the center of disturbance and, in the Northern Hemisphere, is directed around the compass from north to west (the direction of the whirling motion is from north to east in the Southern Hemisphere).

67. The effect of the rotation of the earth upon the motion of a pendulum; Foucault's pendulum. In the discussion of the plane pendulum given in § 45 it was seen that the bob of the pendulum moves in a fixed plane. By "fixed" is meant fixed in the reference frame relative to which there is a constant force mg acting on the bob and in which the point of suspension is fixed. If we choose as our reference frame one whose origin is at the point of suspension and which has no angular velocity relative to an inertial frame, we see that the conditions stated are satisfied for a short interval of time (provided we neglect the infinitesimal changes in the "force of gravity" mg on the bob). Since once every sidereal day the direction of the vertical makes a complete swing around the fixed direction furnished by the axis of rotation of the earth, it is evident that our neglect of the changes in the force of gravity on the bob, which force has the direction of the downward vertical, limits the applicability of the simple theory given in § 45 to an interval of time which is short in comparison with a sidereal day. Since the bob of the pendulum will move in a plane which is fixed relative to the reference frame described, it will not, in general, move in a plane which is fixed relative to the earth. The angular velocity $\boldsymbol{\omega}$, relative to our reference frame, of any frame rigidly attached to the earth may be analyzed into two parts:

1. A component of magnitude $\omega \cos \theta$ along the upward vertical.

2. A component of magnitude $\omega \sin \theta$ along the horizontal line directed toward the north.

A vertical plane fixed in our reference frame will appear, when regarded from the earth, to turn with an angular velocity $\omega \cos \theta$ around the *downward* vertical, so that the direction of rotation in the Northern Hemisphere is from north to east, and in the Southern Hemisphere from north to west. The time taken by the plane of the pendulum to make a complete swing around the compass is $\frac{2\pi}{\omega \cos \theta}$, or $\sec \theta$ sidereal days, since $\frac{2\pi}{\omega}$ is a sidereal day. In terms of the latitude L on the earth, we may say that the plane of vibration of the pendulum makes one complete revolution around the compass in cosec L sidereal days. This is the basis of the theory of Foucault's famous pendulum experiment, which was devised to demonstrate the rotation of the earth relative to an inertial frame.

In order to give a more detailed account of the effect of the rotation of the earth upon the simple pendulum, let us introduce a frame which is attached to the earth and whose origin is at the point of suspension. Taking the upward vertical as our z-axis, the x-axis pointing east and the y-axis north, and denoting the magnitude of the (absolute) tension exerted by the supporting string on the bob by T, our equations of motion are

$$\begin{aligned} \ddot{x} &= -\frac{T}{m}\frac{x}{l} + 2\,\omega(\dot{y}\cos\theta - \dot{z}\sin\theta), \\ \ddot{y} &= -\frac{T}{m}\frac{y}{l} - 2\,\omega\dot{x}\cos\theta, \qquad\qquad (67.1) \\ \ddot{z} &= -\frac{T}{m}\frac{z}{l} - g + 2\,\omega\dot{x}\sin\theta, \end{aligned}$$

where l is the length of the supporting string.

We shall now suppose that the pendulum is executing small oscillations, so that x and y are infinitesimal as compared with l. The equation $x^2 + y^2 + z^2 = l^2$ shows that $z = \pm\, l$ to the first order of infinitesimals; and the equation $x\dot{x} + y\dot{y} + z\dot{z} = 0$ then shows that $\dot{z}$ is an infinitesimal of the second order, since $\dot{x}$ and $\dot{y}$ are infinitesimals of the same order as x and y. We put $z = -\,l$, since we are considering oscillations of the bob near the lowest point of the sphere on which it is constrained to move. A further differentiation gives $x\ddot{x} + y\ddot{y} + z\ddot{z} + (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0$; and as x and y are, by (67.1), infinitesimals of the first order, z is an infinitesimal of the second order. We have, therefore, from the third of the equations (67.1), upon neglecting infinitesimal in comparison with noninfinitesimal quantities, $\frac{T}{m} = g$; so that the first two equations (67.1) may be written in the form

$$\begin{aligned} \ddot{x} &= -\frac{gx}{l} + 2\,\omega\cos\theta\,\dot{y}, \\ \ddot{y} &= -\frac{gy}{l} - 2\,\omega\cos\theta\,\dot{x}. \end{aligned} \qquad (67.2)$$

Upon introducing the complex variable $u = x + iy$, these may be combined into the single equation

$$\ddot{u} = -\frac{gu}{l} - 2\,i\omega\cos\theta\,\dot{u}. \qquad (67.2\,a)$$

This is a linear homogeneous differential equation of the second order whose characteristic equation (see § 38),

$$m^2 + 2\,i\omega\cos\theta\, m + \frac{g}{l} = 0,$$

has pure imaginary roots which we may denote by $i\alpha$ and $-i\beta$, where α and β are both positive numbers of which β is the greater. We have, in fact,

$$\alpha = \left(\frac{g}{l} + \omega^2\cos^2\theta\right)^{\frac{1}{2}} - \omega\cos\theta,\ \ \beta = \left(\frac{g}{l} + \omega^2\cos^2\theta\right)^{\frac{1}{2}} + \omega\cos\theta. \quad (67.3)$$

The general solution of (67.2 a) is therefore $u = Ae^{i\alpha t} + Be^{-i\beta t}$, where A and B are arbitrary complex constants. Let us suppose that the bob starts from rest from a point in the (x, z) plane so that the initial values of u and $\dot{u}$ are C and 0 respectively, where C is a real constant. This gives

$$u = \frac{C}{\alpha + \beta}\,(\beta e^{i\alpha t} + \alpha e^{-i\beta t}), \quad (67.4)$$

or, on separating out the real and imaginary parts,

$$x = \frac{C}{\alpha + \beta}\,(\beta\cos\alpha t + \alpha\cos\beta t),\ \ y = \frac{C}{\alpha + \beta}\,(\beta\sin\alpha t - \alpha\sin\beta t). \quad (67.5)$$

These are the equations of the projection of the bob of the pendulum on a horizontal plane. Upon squaring and adding, we find

$$r = (x^2 + y^2)^{\frac{1}{2}} = \frac{C}{\alpha + \beta}\,[\beta^2 + \alpha^2 + 2\,\alpha\beta\cos(\alpha + \beta)t]^{\frac{1}{2}}. \quad (67.6)$$

This tells us that the maximum value of r is C and the minimum value $\frac{C(\beta - \alpha)}{\beta + \alpha}$, the maximum value being attained when $(\alpha + \beta)t$ is an even multiple of π, and the minimum value when $(\alpha + \beta)t$ is an odd multiple of π. In other words, the projection of the bob on a horizontal plane lies in the ring between two circles of radii C and $\frac{C\omega\cos\theta}{\left[\frac{g}{l} + \omega^2\cos^2\theta\right]^{\frac{1}{2}}}$, the common center of the two circles lying vertically under the point of suspension.

It follows from (67.5) that $\dot{x}$ and $\dot{y}$ vanish together when $(\alpha + \beta)t$ is an even multiple of π, at which times the projection of the bob is on the outer circle of the ring. Hence the projection has stationary points, commonly called cusps, at its points of intersection with the outer circle $r = C$, and the time interval between two cusps is $\frac{2\,\pi}{\alpha + \beta}$. These cusps correspond to the end points of the swing in the more elementary theory, and between them the path of the projection is practically a straight line. In fact, it follows from (67.5) that

$\frac{dy}{dx} = \frac{-\tan(\beta - \alpha)t}{2}$, so that the inclination of the tangent to the projection of the bob to the x-axis is either $-\omega \cos \theta . t$ or $\pi - \omega \cos \theta . t$. Since ω is less than 10^{-4}, by (65.2), this inclination hardly varies during the time of a half-swing. The period of a complete swing is $\frac{4\pi}{\alpha + \beta}$ or $\frac{2\pi}{\left(\frac{g}{l} + \omega^2 \cos^2 \theta\right)^{\frac{1}{2}}}$, so that one effect of the rotation of the earth is a very slight decrease in the period. This diminution of the period is quite negligible, since it is of the order of magnitude of ω^2.

When $t = \frac{2n\pi}{\alpha + \beta}$, so that the projection of the bob is at one of the cusps on the outer circle, we see, from (67.4), which may be written in the form $u = \frac{Ce^{i\alpha t}}{\alpha + \beta}[\beta + \alpha e^{-i(\alpha + \beta)t}]$, that the argument of the complex quantity u is αt. The argument of the next cusp but one — that is, the cusp reached after a complete swing — is greater than this by $\frac{4\pi\alpha}{\alpha + \beta}$, since the period of a complete swing is $\frac{4\pi}{\alpha + \beta}$. Since α and β are very nearly equal, β being very slightly greater than α, this says that during a complete swing there is an *increase* in the argument of the cusp of amount slightly less than 2π. An equivalent way of saying this is that there is a slight *decrease* in the argument of the cusp. The amount of this decrease is

$$2\pi - \frac{2\pi\alpha}{\left(\frac{g}{l} + \omega^2 \cos^2 \theta\right)^{\frac{1}{2}}}, \quad \text{or} \quad \frac{2\pi\omega \cos \theta}{\left(\frac{g}{l} + \omega^2 \cos^2 \theta\right)^{\frac{1}{2}}}. \qquad (67.7)$$

In each complete swing of the pendulum its plane turns, therefore, from north to east, through this angle. Since $\omega \cos \theta$ is very small, l should be made as large as possible if we wish to observe this steady rotation of the plane of vibration of the pendulum. The most accurate determination of ω, by means of the result (67.7), is that due to Kamerlingh Onnes. The accuracy of the determination of the length of the day was within 8 minutes, that is, about one part in 200.

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EXERCISES

1. Prove that the axis of a top whose principal moments of inertia are (A, A, C) and whose spin is ν will keep step with that of a top whose moments are each equal to A and whose spin is $\frac{C\nu}{A}$, provided the latter top be suitably started. Show also that the Eulerian angles ψ and ψ' in the two cases are connected by the relation

$$\psi' = \psi + \frac{(C - A)}{A}\,\nu t.$$

2. The earth's mass being assumed to be 6×10^{21} tons, find the change in the length of the day if icebergs and melted snow weighing 10^{10} tons move from the north pole to latitude 45 degrees.

3. Is there a change in the displacement of a vessel running east or west at the rate of 30 miles per hour? Why, and how much?

4. If the sun gradually contracts in such a way as to remain always similar to itself in constitution and form, show that when every radius has contracted an nth part of its length, where n is large, the angular velocity will have increased to $\left(1 + \frac{2}{n}\right)$ times its former value. Examine the change in the kinetic energy of the rotation.

5. Discuss the motion of a uniform rod under gravity about a horizontal axis through one end. Treat also the conical pendulum motion of the rod about one of its ends, which is fixed.

6. Discuss the motion of any rigid body about a horizontal axis, the resistance of the air being neglected. Show that the length of the simple equivalent pendulum is given by the formula $l = \frac{k^2 + d^2}{d}$, where Mk^2 is the moment of inertia of the body about a parallel axis through its center of mass, and d is the distance between this axis and the axis of rotation. Hence show that the distance between two horizontal axes which are coplanar with the center of mass and such that the period of the oscillations of the body is the same for each, is equal to the length of the simple equivalent pendulum. If the periods are not exactly equal, and if the distances of the two axes of rotation from the center of mass are d_1 and d_2 respectively, show that $\frac{8\pi^2}{g} = \frac{T_1^2 + T_2^2}{d_1 + d_2} + \frac{T_1^2 - T_2^2}{d_1 - d_2}$. This result furnishes a reliable method for calculating the value of g.

7. If a right circular cone, whose altitude a is double the radius of its base, turns about its center of gravity as a fixed point, and is originally set in motion about an axis inclined at an angle α to the axis of figure, the vertex of the cone will describe a circle whose radius is $\frac{3}{4} a \sin \alpha$.

8. A solid ellipsoid, the squares of whose semiaxes are c^2, $3c^2$, $5c^2$, is set rotating about a diameter lying in the plane of the greatest and least axes and making an angle whose cotangent is $(2)^{\frac{1}{2}}$ with the former. Find the initial direction cosines of the invariable line referred to the axes of the ellipsoid, and show that the angular velocity of the mean axis about the invariable line is constant during the subsequent motion.

CHAPTER X

GYROSCOPIC THEORY

68. The effect of an impulse applied to the axis of figure of a gyroscope. In 1852 Foucault showed that a dynamically symmetrical body spinning around its axis of figure could be used to demonstrate the rotation of the earth with respect to an inertial frame, and he named such a spinning body a gyroscope. We have already seen that if the body is balanced about a fixed point on its axis of figure and is started spinning around this axis, it will continue to rotate with constant angular velocity around the axis of figure until it is acted upon by some external force having a moment about the point of suspension. Let us now examine qualitatively the effect of an impulse which intersects the axis of figure but does not pass through the point of suspension. The applied impulse is equivalent to an equal impulse through the point of suspension, together with an impulse moment about this point.

In discussing the motion of the body about the point of support, we may confine our attention to the impulse moment about the point of support, the impulse through the point of support being neutralized by an impulsive reaction there. The fact that we consider the impulse *moment* and *not* the applied impulse explains many of the apparently puzzling paradoxes of gyroscopic theory. The fundamental theorem that the change in angular momentum of the body about the point of support is equal to the moment of the applied impulse about this point enables us to calculate the new angular-momentum line, that is, the new invariable line; and from a knowledge of this and the principal moments of inertia about the fixed point the new position of the rotation vector $\boldsymbol{\omega}$ follows. There is no abrupt change in the position of the axis of figure, since, under an impulse, the velocities and not the positions of the various points of the body undergo a sudden change. The moment of the impulse is perpendicular to the axis of figure and is $Ap\mathbf{i} + Aq\mathbf{j}$, where (**i**, **j**, **k**) are unit vectors, mutually perpendicular, the third of which lies along the axis of figure about which the body was rotating before the application of the impulse. The new rotation vector is $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$, the original rotation vector having been $r\mathbf{k}$; so that the change in the rotation vector is $p\mathbf{i} + q\mathbf{j}$, whereas the change in the angular-momentum vector is A times this. After the application of the impulse, the axis of rotation no longer coincides with the invariable line, as it did before the application of the impulse; and we have the rule

The axis of rotation always moves in the direction of the moment of the applied impulse.

If the magnitude of this moment is M, the angle through which the invariable line moves is the arc tan $\frac{M}{Cr}$, and the angle through which the axis of rotation moves is the arc tan $\frac{M}{Ar}$; for $M = A(p^2 + q^2)^{\frac{1}{2}}$. If r is very large, the angle through which both the axis of rotation and the invariable line move is very small. This explains the stability of a rapidly spinning gyroscope with respect to impulses. Nothing can be inferred from this, however, as to its stability with respect to continuous forces.

Immediately after the application of the impulse, the body begins to execute a steady precession about the new invariable line. This precessional motion is direct or retrograde according as A is greater or less than C.

69. The forced precessional motion of the axis of figure. We shall now consider the case of a dynamically symmetrical body with a fixed point, acted on by forces having a moment about the point of support. It will be of interest to find out what applied moment is necessary to make the axis of figure execute a steady precessional motion, that is, to make the axis of figure move with a given constant angular velocity $\dot{\phi} = \mu$ about a given fixed line, while at the same time the body has a constant velocity of rotation $\dot{\psi} = \nu$ about its axis of figure. Taking the fixed line as the $O\zeta$ axis of a fixed frame, we shall use the Eulerian angles (θ, ϕ, ψ) of § 23. The components of the angular velocity $\boldsymbol{\omega}$ are μ along the $O\zeta$ axis and ν along the OZ axis, so that the components in the moving frame $OXYZ$ are, from the table (23.9),

$$p = \mu \sin\theta \cos\psi, \quad q = \mu \sin\theta \sin\psi, \quad r = \mu\cos\theta + \nu. \qquad (69.1)$$

We now wish to find out what applied moment is necessary to keep the three quantities (θ, μ, ν) constant in these expressions. The answer is furnished at once by the Eulerian equations (§ 58). Upon substituting in the Eulerian equations the derivatives

$$\frac{dp}{dt} = \mu\nu \sin\theta \sin\psi, \quad \frac{dq}{dt} = \mu\nu \sin\theta \cos\psi, \quad \frac{dr}{dt} = 0, \tag{69.2}$$

(in deriving which we have used the equation of definition $\dot{\psi} = \nu$), we obtain for the components (L, M, N) of the necessary applied moment, in the moving frame, the results

$$\begin{aligned} L &= A\frac{dp}{dt} - (A - C)qr \\ &= C\mu\nu \sin\theta \sin\psi - (A - C)\mu^2 \sin\theta \cos\theta \sin\psi, \\ M &= A\frac{dq}{dt} - (C - A)rp \\ &= C\mu\nu \sin\theta \cos\psi - (A - C)\mu^2 \sin\theta \cos\theta \cos\psi, \\ N &= C\frac{dr}{dt} = 0. \end{aligned} \tag{69.3}$$

Since the direction cosines of the line of nodes, relative to the moving frame, are ($\sin\psi$, $\cos\psi$, 0), we see, from (69.3), that the applied moment which is necessary to maintain the steady precession with angle θ is directed along or opposite to the line of nodes and has the magnitude

$$\pm \mu \sin\theta [C\nu + (C - A)\mu \cos\theta]. \tag{69.4}$$

This result is one of the apparent paradoxes of the theory. It is, at first sight, somewhat surprising that the applied moment necessary to maintain θ constant is exactly in the direction which would make it change θ if the body were at rest. The situation is, however, exactly analogous to the case of uniform circular motion of a particle where the applied force necessary to maintain r constant is exactly in the direction which would make it decrease r if the particle were at rest. Just as we speak of the centrifugal, or inertial, force of the particle moving in the circle, — this being the *negative* of the applied force necessary to maintain the uniform circular motion, — so we here speak of the deviation, or inertial, moment of the gyroscope which is executing the uniform forced precession, this being the negative of the applied moment necessary to maintain the uniform precession. Denoting by $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ the vectors, directed along the invariable line and the axis of figure respectively, whose magnitudes are $\dot{\phi}$ and $\dot{\psi}$ respectively, and whose sum is the angular velocity vector $\boldsymbol{\omega}$, we may write the inertial or deviation moment of the gyroscope in the form

$$\mathbf{D} = \left[C + (C - A)\frac{\mu}{\nu}\cos\theta\right][\boldsymbol{\nu}.\boldsymbol{\mu}]; \tag{69.5}$$

for the magnitude of the vector product $[\boldsymbol{\nu}.\boldsymbol{\mu}]$ is $\mu\nu\sin\theta$, and the direction of this vector product is opposite to that of the line of nodes. The single expression (69.5) will answer all the more elementary questions as to the behavior of a gyroscope. If the applied moment necessary to maintain the forced precession is not forthcoming, the axis of figure of the gyroscope will begin to move in the direction (of increase or decrease of θ) in which it would move, if initially at rest, under the action of an applied moment **D**. Just as in the case of the particle moving uniformly in a circle, the particle will tend to fly away from the center when the central force necessary to maintain the uniform circular motion is lacking, and this would be the result of applying the centrifugal force to a particle at rest.

If the gyroscope is spinning rapidly, we may usually neglect the precessional velocity μ in comparison with the velocity of spin ν, and the deviation moment takes the very simple form

$$\mathbf{D}_1 = C[\boldsymbol{\nu}.\boldsymbol{\mu}]. \tag{69.6}$$

This is exactly what **D** would be if $C = A$, that is, if the point of support were a spherical point of the gyroscope; and it is for this reason called the spherical part of the deviation moment. The remaining part,

$$\mathbf{D}_2 = \frac{(C - A)\mu\cos\theta}{\nu}[\boldsymbol{\nu}.\boldsymbol{\mu}], \tag{69.7}$$

of the deviation moment is called the ellipsoidal part. It will be observed that the magnitude of $\mathbf{D}_2$ is independent of the magnitude ν of the velocity of spin, and it is, in fact, what **D** would reduce to if ν were zero. It follows, from (69.6), that if there is no applied moment, the axis of a rapidly spinning gyroscope will tend to move so as to decrease θ; that is, the tendency of the axis of figure is to move into coincidence with the invariable line about which it would precess if the necessary moment were supplied. For a dynamically symmetrical body moving as a conical pendulum (that is, without any angular velocity about its axis of figure) the tendency of the axis of figure is to move toward or away from the invariable line according as C is greater or less than A; for then the deviation moment is given by (69.7). (Since there is no angular velocity about the axis of figure, we are free to choose either of the two directions along the principal axis of inertia of mean moment at the point of support as the direction of our axis of figure, and we avail ourselves of this freedom of choice to arrange matters so that the angle θ is not obtuse.)

If we give up the hypothesis that the angle θ is constant, we find at once, on substituting in Euler's equations the expressions

$$\begin{aligned} p &= \dot{\theta} \sin \psi - \sin \theta \cos \psi \, \dot{\phi}, \\ q &= \dot{\theta} \cos \psi + \sin \theta \sin \psi \, \dot{\phi}, \\ r &= \dot{\psi} + \cos \theta \, \dot{\phi}, \end{aligned}$$

given in (26.10), and resolving the resulting vector equation along the line of nodes, that

$$A\ddot{\theta} = (\text{magnitude of deviation moment}) + (\text{resolved part of applied moment along line of nodes}). \qquad (69.8)$$

The introduction of the deviation moment is therefore the same as a reduction of the gyroscope to rest, as far as θ is concerned. To find the nutation θ of the gyroscope, we may introduce the deviation moment and then proceed as if the gyroscope were executing a simple rigid pendulum motion about the line of nodes.

70. The various types of forced uniform precessional motion; the pressure between the polhode cones. Since $\boldsymbol{\omega}$ is the sum of the two vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, which are inclined to each other at an angle θ, we have

$$\omega^2 = p^2 + q^2 + r^2 = \mu^2 + \nu^2 + 2\,\mu\nu \cos \theta,$$

so that ω is constant. Moreover, $r = \mu \cos \theta + \nu$ is constant, and so $p^2 + q^2 = \mu^2 \sin^2 \theta$ is constant. Since $2\,T = A(p^2 + q^2) + Cr^2$ and $G^2 = A^2(p^2 + q^2) + C^2r^2$, this shows that the kinetic energy T and the magnitude G of the angular-momentum vector are constants. The difference between the forced uniform precession and the free Poinsot precession under no moments is that in the former the direction of G is not fixed, whereas in the latter it is. In the moving reference frame formed by the principal axes of inertia at the point of support, the components of the various vectors involved are

$$\begin{aligned}
&\boldsymbol{\omega} = (p, q, r) = (-\mu \sin\theta \cos\psi,\ \mu \sin\theta \sin\psi,\ \mu\cos\theta + \nu),\\
&\mathbf{G} = (Ap,\ Aq,\ Cr)\\
&\quad = (-A\mu \sin\theta \cos\psi,\ A\mu \sin\theta \sin\psi,\ C\mu \cos\theta + C\nu),\\
&\text{Axis of figure} = (0, 0, 1), \qquad\qquad (70.1)\\
&\text{Axis of precession} = (-\sin\theta\cos\psi,\ \sin\theta\sin\psi,\ \cos\theta),\\
&\text{Line of nodes} = (\sin\psi,\ \cos\psi,\ 0).
\end{aligned}$$

These expressions show that the axis of rotation, the axis of angular momentum, the axis of figure, and the axis of precession are all perpendicular to the line of nodes. The *transverse* plane containing these four axes is known as the plane of precession. Furthermore, the angle between any two of the four axes remains constant throughout the precessional motion. Hence, if we draw, at each instant, the invariable plane of a free Poinsot precession determined by the conditions holding at that instant, the various invariable planes thus obtained will envelope a right circular cone, with its vertex on the axis of precession. In fact, the distance $\frac{2T}{G}$ of these invariable planes from the point of support is the same for all the invariable planes, and each invariable plane is perpendicular to the axis of angular momentum (which axis makes a constant angle with the fixed axis of precession). The motion of the rigid body when executing the forced uniform precession may, then, be conveniently described geometrically by the statement that an ellipsoid of revolution fixed in the body (the Poinsot ellipsoid, in fact) rolls upon a fixed right circular cone whose axis is the axis of precession. If we try to realize this geometric description mechanically, the applied moment necessary to maintain the forced uniform precession may be regarded as furnished by the reaction exerted by the fixed cone upon the moving spheroid. It is, however, more convenient to have recourse to the geometric description of the motion in terms of the rolling of the polhode cone upon the herpolhode cone. The two cones are right circular cones in the present instance, and we may regard the necessary applied moment as furnished by the reaction of the herpolhode cone upon the polhode cone.

In addition to the two types of precessional motion considered in § 64, there is now a third type, since there is no longer any necessity for the angle between the axis of precession and the axis of rotation to be acute. (In the free Poinsot precession this angle has to be acute, since the axis of precession coincides there with the axis of angular momentum.) The cosine of the angle between the axis of precession and the axis of rotation is $\frac{(\mu + \nu \cos \theta)}{\omega}$. The new type of precession will take place when this is negative, so that the angle between the axis of precession and the axis of rotation is obtuse. Denote this angle by $\pi - \epsilon$, so that ϵ is an acute angle, the semivertical opening of the herpolhode cone. Since $\boldsymbol{\omega}$ is the sum of the two vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, the axis of rotation must lie between the axis of precession and the axis of figure, which tells us that θ is also an obtuse angle and greater than $\pi - \epsilon$. The semivertical opening of the polhode cone is the difference between these two obtuse angles; that is,

$$\delta = \theta - (\pi - \epsilon). \tag{70.2}$$

As this is less than ϵ, the polhode cone is enveloped by the herpolhode cone on which it rolls internally. For this reason the precession is said to be of the hypocycloidal type.

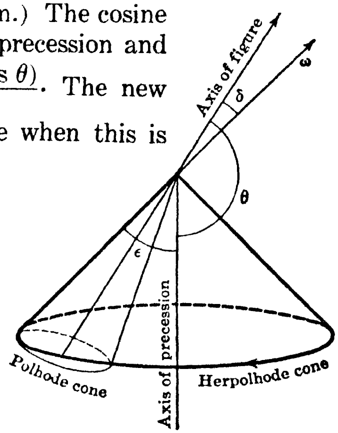

FIG. 31

It may be observed that the two types of precession, direct and retrograde, already described in § 64, may occur in a forced precession quite independently of the relative magnitude of the moments of inertia A and C, which was the determining factor as to which type of precession would occur in the Poinsot free precession. If the necessary applied moment is supplied, a gyroscope may be forced to describe either type of precessional motion.

It is not difficult to express the applied moment, which is necessary to maintain the forced precession, in terms of the openings of the two cones, the velocity of spin, and the angle of precession θ. Since ω is the resultant of μ and ν, we have $\mu \sin \epsilon = \nu \sin \delta$, so that the magnitude of the applied moment, which lies along or opposite to the line of nodes, is, by (69.4), the numeric value of

$$\frac{\nu \sin \delta \sin \theta}{\sin \epsilon}\left[C\nu + \frac{(C - A)\nu \sin \delta \cos \theta}{\sin \epsilon}\right]. \tag{70.3}$$

When the expression $C \sin \epsilon + (C - A) \sin \delta \cos \theta$ is negative, the deviation moment, which is the negative of the applied moment, is directed along the line of nodes and tends to increase θ. In other words, if $C \sin \epsilon + (C - A) \sin \delta \cos \theta$ is negative, the gyroscope, in the absence of the necessary applied moment, moves in such a way as to separate the polhode cone from the herpolhode cone in all three types of precession. Conversely, if this expression is positive, the gyroscope has, in the absence of an applied moment, a tendency to decrease θ, that is, to press the polhode cone against the herpolhode cone. In this event the reaction of the herpolhode cone will supply the necessary moment to maintain the precession. This implies, of course, that the motion is realized mechanically by the rolling of one material right circular cone upon another.

It is easy to see that there is always a pressure between the two cones in the hypocycloidal case. For then, by (70.2), $\epsilon = \pi + \delta - \theta$, and the expression

$$C \sin \epsilon + (C - A) \sin \delta \cos \theta = C \sin \theta \cos \delta - A \sin \delta \cos \theta$$

is positive, since $\cos \theta$ is negative, while the other terms are positive. In the epicycloidal case $\theta = \delta + \epsilon$, by (64.12). Consequently the condition for pressure between the two cones is

$$C \sin \epsilon + (C - A) \sin \delta \cos (\delta + \epsilon) > 0,$$

or

$$\tan \epsilon > \frac{(A - C) \tan \delta}{C + A \tan^2 \delta}. \tag{70.4}$$

There will therefore always be a pressure between the cones in this precession if A is not greater than C; whereas if A is greater than C, the semivertical opening of the herpolhode cone must be greater than the angle whose tangent is $\dfrac{(A - C) \tan \delta}{C + A \tan^2 \delta}$ if there is to be pressure between the two cones.

In the pericycloidal case $\theta = \pi + \epsilon - \delta$, by (64.13); and the condition for pressure may be put in the form

$$C \sin \theta \cos \delta + A \sin \delta \cos \theta < 0,$$

or

$$C \tan \theta + A \tan \delta > 0, \tag{70.5}$$

since $\cos \theta$ is negative whereas $\cos \delta$ is positive.

In the realization in practice of the forced precession the semi-vertical opening δ of the polhode cone is very small, so that the polhode cone does not differ very sensibly from a cylinder. In the pericycloidal case the opening of the herpolhode cone is smaller than that of the polhode cone, and this type is not convenient to construct. In the other two cases, however, the opening of the polhode cone may be made as small as we please; and this polhode cone may, in fact, be sufficiently approximated by the material axis of figure of the gyroscope, even if this axis is cylindrical rather than conical in shape. The axis of figure may follow a "guide" whose projection from the fixed point will be the herpolhode cone. We may regard the actual motion at any instant as sufficiently approximated by the "tangent uniform forced precession," that is, by the forced precession for which the herpolhode cone would be the right circular cone that has contact with the actual herpolhode cone along a generator, and for which μ and ν have the values given by the actual motion at the instant in question. The axis of figure will adhere to the guide, no matter what its shape, as long as the tangent uniform precession is of the hypocycloidal type, that is, when the guide is concave to the axis. It will also adhere to the guide even in the epicycloidal case, that is, when the guide is convex to the axis of figure, provided the condition of (70.4) is satisfied. This is the explanation of the "clinging" of a gyroscope to a guide along which its axis of figure rolls. If the axis leaves the guide, the necessary moment to maintain the forced precession is no longer supplied, and the gyroscope will begin a free Poinsot precession about the angular-momentum line. It is possible that the axis may leave the guide and again come into contact with it during its free precessional motion, in which event it would again take up what may be regarded as a sequence of forced uniform precessional motions.

Advantage may be taken of the pressure between the polhode and herpolhode cones, which is always present in the hypocycloidal precession, to construct a grinding mill. The magnitude of the pressure is seen, from (70.3), to be proportional to ν^2, so that high pressures may be developed if the gyroscope is spinning rapidly.

71. The motion of a dynamically symmetrical body about a fixed point under the influence of its own weight. We have supposed heretofore either that the weight of the gyroscope was negligible or that the gyroscope was balanced, that is, suspended with its center of mass at the point of support so that its weight had no moment about this point. We shall now assume that the center of mass, while still located on the axis of figure, is not at the point of support. Let us take as the $O\zeta$ axis of our fixed frame the upward vertical and as the OZ axis of our moving frame the axis of figure directed *from* the point of support, O, *to* the center of mass. Then the applied moment (here the moment of the weight of the gyroscope) is directed along the line of nodes and has the components $(Mgl \sin \theta \sin \psi,\ Mgl \sin \theta \cos \psi,\ 0)$ in the moving frame, where M is the mass of the gyroscope and l is the distance from the point of support to the center of mass. (These components follow at once from the observation that the moment wanted is the vector product of the two vectors $(0,\ 0,\ 1)$ and $(Mg \sin \theta \cos \psi,\ -Mg \sin \theta \sin \psi,\ -Mg \cos \theta)$.) On using, as before, $(p,\ q,\ r)$ to denote the components in the moving frame of the angular-velocity vector $\boldsymbol{\omega}$, the Eulerian equations (58.2) become

$$\begin{aligned} A\frac{dp}{dt} - (A - C)qr &= Mgl \sin \theta \sin \psi, \\ A\frac{dq}{dt} - (C - A)rp &= Mgl \sin \theta \cos \psi, \\ C\frac{dr}{dt} &= 0, \end{aligned} \tag{71.1}$$

the last of which gives $r = n$, where n is a constant. A second integral of these equations is a consequence of the fact that the time derivative of the angular-momentum vector **G**, being equal to the applied moment, has its resolved part along the fixed line $O\zeta$ equal to zero. Hence the resolved part G_ζ of **G** along the upward vertical is constant, giving

$$-Ap \sin \theta \cos \psi + Aq \sin \theta \sin \psi + Cn \cos \theta = G_\zeta. \tag{71.2}$$

On multiplying equations (71.1) by (p, q, r) respectively and adding, we obtain, after using the relation $p \sin \psi + q \cos \psi = \dot{\theta}$, which follows from (26.10), the energy integral

$$A(p^2 + q^2) + Cn^2 = -2\,Mgl \cos \theta + 2\,h, \qquad (71.3)$$

where h is a constant giving the total energy of the gyroscope. (Of course, the energy integral is best written down directly, independently of the Eulerian equations.) Upon making the substitutions of (26.10),

$$\begin{aligned} p &= \dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi, \\ q &= \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi, \\ r &= n = \dot{\psi} + \dot{\phi} \cos \theta, \end{aligned}$$

our integrals (71.2) and (71.3) take the forms

$$A \sin^2 \theta\, \dot{\phi} + Cn \cos \theta = G_\zeta, \qquad (71.2\,a)$$

$$A(\dot{\theta}^2 + \sin^2 \theta\, \dot{\phi}^2) + Cn^2 = 2(h - Mgl \cos \theta). \qquad (71.3\,a)$$

We notice, from (71.2 *a*), that $\sin \theta$ cannot become zero during the motion — that is, the axis of figure cannot come into coincidence with the upward or downward vertical — unless the initial conditions are such that

$$G_\zeta = \pm\, Cn. \qquad (71.4)$$

We shall reserve for a special discussion the case where this relation between the constants of the problem exists, and we proceed to deal with the general case in which the axis of figure cannot become vertical. The angular velocities $\dot{\phi}$ and $\dot{\psi}$ may be expressed in terms of the cosine of the angle θ by the equations

$$\dot{\phi} = \frac{G_\zeta - Cn \cos \theta}{A(1 - \cos^2 \theta)}, \qquad (71.5)$$

$$\dot{\psi} = n - \dot{\phi} \cos \theta = Cn\left(\frac{1}{C} - \frac{1}{A}\right) + \frac{Cn - G_\zeta \cos \theta}{A(1 - \cos^2 \theta)}. \qquad (71.6)$$

Adopting the notation $\cos \theta = \zeta$, ζ will be the vertical distance above (or below, if negative) the point of support of a point on the axis of figure at unit distance from O in the direction from O to the center of mass. This point is called the vertex of the heavy gyroscope or top. Upon eliminating $\dot{\phi}$ from (71.3 a) by means of (71.5), we find

$$\left(\frac{d\zeta}{dt}\right)^2 = (\alpha - a\zeta)(1 - \zeta^2) - (\beta - bn\zeta)^2 = f(\zeta), \text{ say,} \tag{71.7}$$

where we have adopted the abbreviations

$$\alpha = \frac{2\,h - Cn^2}{A}, \quad \beta = \frac{G_\zeta}{A}, \quad a = \frac{2\,Mgl}{A}, \quad b = \frac{C}{A}. \tag{71.8}$$

Our assumption that the top cannot become vertical implies that β is not equal to $\pm\, bn$. Once we have solved the differential equation (71.7) for ζ as a function of t, we have to integrate the equations

$$\dot{\phi} = \frac{\beta - bn\zeta}{1 - \zeta^2}, \tag{71.5 a}$$

$$\dot{\psi} = n(1 - b) + \frac{bn - \beta\zeta}{1 - \zeta^2}, \tag{71.6 a}$$

in order to obtain ϕ and ψ as functions of t.

The polynomial $f(\zeta)$, defined by (71.7), is a cubic in ζ with three real zeros. For if ζ_0 is the initial value of ζ, $f(\zeta_0)$ must be either positive or zero, since it is the initial value of $\left(\frac{d\zeta}{dt}\right)^2$, while $f(\pm 1) = -\,(\beta \mp bn)^2$ are both definitely negative. Moroever, $f(\zeta)$ is positive when ζ is large and positive, since the coefficient of the ζ^3 term is positive. Denoting the three real zeros of $f(\zeta)$, when arranged in descending order of magnitude, by $(\zeta_1, \zeta_2, \zeta_3)$ respectively, we have

$$\zeta_1 > 1 > \zeta_2 \geqq \zeta_0 \geqq \zeta_3 > -1.$$

We are now in a situation which has already been discussed in detail in § 46. The differential equation (71.7) may be written in the form

$$\left(\frac{d\zeta}{dt}\right)^2 = a(\zeta_1 - \zeta)(\zeta_2 - \zeta)(\zeta - \zeta_3),$$

and ζ must lie in the interval $\zeta_2 \geqq \zeta \geqq \zeta_3$. In order to reduce our differential equation to the standard form

$$\left(\frac{dp}{dt}\right)^2 = 4(e_1 - p)(e_2 - p)(p - e_3), \quad (e_1 + e_2 + e_3 = 0)$$

we write

$$\begin{aligned} \zeta &= \frac{4\,p}{a} + \frac{\alpha + b^2n^2}{3\,a}, \\ t &= u + \beta, \quad (\beta = \text{any constant}) \end{aligned} \tag{71.9}$$

In terms of our original constants we find, on comparing this with (71.8),

$$\zeta = \frac{2\,Ap(u)}{Mgl} + \frac{2\,hA - C(A - C)n^2}{6\,AMgl}, \qquad (71.9\,a)$$

$$t = u - \omega_3,$$

where ω_3 is the purely imaginary semiperiod of the Weierstrassian elliptic function $p(u)$ (see § 46). Now that the equation (71.7) has been reduced to the standard form, the work is precisely as in § 46. We have, on writing

$$\eta = \text{arc sin}\left(\frac{\zeta - \zeta_3}{\zeta_2 - \zeta_3}\right),$$

$$t = \frac{F(\eta, k)}{(e_1 - e_3)^{\frac{1}{2}}} = \left(\frac{2\,A}{Mgl}\right)^{\frac{1}{2}} F\,\frac{(\eta, k)}{(\zeta_1 - \zeta_3)^{\frac{1}{2}}}, \qquad (71.10)$$

where F is Legendre's normal integral of the first kind, and the modulus k is $\left(\frac{\zeta_2 - \zeta_3}{\zeta_1 - \zeta_3}\right)^{\frac{1}{2}}$. In using this result, time is measured from an instant at which the vertex of the top is at its lowest point $\zeta = \zeta_3$. The vertical motion of the axis of figure is periodic, the period being

$$T = \left(\frac{2\,A}{Mgl}\right)^{\frac{1}{2}} \frac{4\,K}{(\zeta_1 - \zeta_3)^{\frac{1}{2}}},$$

where K is the complete elliptic integral.

Since ζ is periodic, it follows, from (71.5 a) and (71.6 a), that the angular velocities $\dot{\phi}$ and $\dot{\psi}$ are also periodic with the same period as ζ.

72. Discussion of the spherical path of the heavy gyroscope or top. The vertex of the top traces out a path on the sphere of unit radius whose center is at the point of support O. At any point of this path whose latitude is L and whose longitude is λ the tangent to the path makes with the parallel of latitude an angle whose tangent is $\sec L \frac{dL}{d\lambda}$ (since the element of arc along a meridian is simply dL, whereas the element of arc along a parallel of latitude is $\cos L\,d\lambda$). In terms of $\zeta = \sin L$ we see that i, the inclination of the path of the vertex to the parallel of latitude through the point in question, is given by the equation

$$\tan^2 i = \frac{f(\zeta)}{(\beta - bn\zeta)^2}.$$

We are using the fixed line $O\zeta$ as the polar axis of our sphere, so that $L = \frac{\pi}{2} - \theta$ and $\lambda = \phi$, and we have made use of the equations (71.7) and (71.5 a). We see, then, that the inclination i will, in general, be zero at points where $f(\zeta) = 0$, that is, at the bounding circles of latitude $\zeta = \zeta_2$ and $\zeta = \zeta_3$. The only exception is when the denominator $(\beta - bn\zeta)^2$ of the expression for $\tan^2 i$ also vanishes at either of these bounding circles. In this exceptional case the path of the vertex, instead of touching the corresponding bounding circle, cuts it at right angles, since the denominator is squared (and so vanishes to a higher order than the numerator). Since both $\dot{\theta}$ and $\dot{\phi}$ are zero at these points, the path of the vertex has cusps at them. It is easy to see that the cusps cannot occur on the lower bounding circle; for at a cusp the kinetic energy of the top has its minimum value $\frac{Cn^2}{2}$, so that the potential energy must have its maximum value. In other words, the vertex of the top must be at its highest point.

From the formula (71.5 a) we see that the horizontal motion of the vertex is always in the same sense unless $\frac{\beta}{bn}$ has a value between ζ_2 and ζ_3; in which case the path of the vertex has loops, since $\dot{\phi}$ changes sign on the circle of latitude $\zeta = \frac{\beta}{bn}$. The condition which the constants of the problem must fulfill so that the path of the vertex may have loops is readily found by making $f\left(\frac{\beta}{bn}\right)$ positive, $\frac{\beta}{bn}$ being less than unity in absolute value. This gives

$$\frac{2h - Cn^2}{2Mgl} > \frac{G_\zeta}{Cn}. \quad \left(-1 < \frac{G_\zeta}{Cn} < 1\right) \tag{72.1}$$

Let us now assume that the position of one of the bounding circles is known, and let us discuss the possibilities for the other bounding circle. We may, for example, suppose that the gyroscope is started off with $\dot{\theta} = 0$ at a given θ. We shall denote the known value of ζ by ζ_0, so that ζ_0 may stand, indifferently, for either ζ_2 or ζ_3. We have, from (71.7),

$$\frac{f(\zeta)}{1 - \zeta^2} = \alpha - a\zeta - \frac{(\beta - bn\zeta)^2}{1 - \zeta^2};$$

and since ζ_0 is a zero of $f(\zeta)$, we have

$$0 = \alpha - a\zeta_0 - \frac{(\beta - bn\zeta_0)^2}{1 - \zeta_0^2}.$$

Upon subtracting these two equations and dividing the result by $(\zeta - \zeta_0)$, we find that the other two zeros of the cubic polynomial $f(\zeta)$ are given by the quadratic equation

$$2\,b\beta n(1 + \zeta\zeta_0) - (\beta^2 + b^2n^2)(\zeta + \zeta_0) \\ - a(1 - \zeta_0^2)(1 - \zeta^2) = 0. \qquad (72.2)$$

For a given value of spin n this gives the relation between β and the ζ of the second bounding circle. This relationship may best be studied by making a graph, using ζ and β as rectangular Cartesian coördinates. The graph is a cubic curve which is met by a parallel to either coördinate axis in two points at most. Thus the vertical lines $\zeta =$ constant meet the graph in the points whose ordinates are given by

$$\beta(\zeta + \zeta_0) = bn(1 + \zeta\zeta_0) \\ \pm\{(1 - \zeta^2)(1 - \zeta_0^2)[b^2n^2 - a(\zeta + \zeta_0)]\}^{\frac{1}{2}}. \qquad (72.3)$$

We have vertical tangents when the two values of β furnished by this equation are equal, that is, when $\zeta = \pm 1$ and when

$$\zeta = -\zeta_0 + \frac{b^2n^2}{a}, \qquad (72.4)$$

the corresponding values of β being $\pm\, bn$ and $\dfrac{a(1 - \zeta_0^2)}{bn} + bn\zeta_0$ respectively. The particular vertical line $\zeta = -\zeta_0$ is exceptional in so far as it meets the curve, as is seen from (72.2), in only one point,

$$\beta = \frac{a(1 - \zeta_0^2)}{2\,bn}. \qquad (72.5)$$

It is at once apparent from (72.3) that there is an imaginary region of the curve in the interval $-1 < \zeta < 1$, to which we are limited by the physical nature of the problem and by the assumption that β is not equal to either $\pm\, bn$, provided that

$$\frac{b^2n^2}{a} - \zeta_0 < 1. \qquad (72.6)$$

In this event the part of the sphere for which

$$\frac{b^2n^2}{a} - \zeta_0 < \zeta < 1$$

cannot be reached by the axis of figure, no matter what value is given to β. The gyroscope or top is said to be *weak* in this case. If, on the other hand, $\dfrac{b^2n^2}{a} - \zeta_0 \geqq 1$, any circle on the sphere can be reached as the second bounding circle. The value to be assigned to β is then determined by (72.3). Two distinct values of β may be taken in order to reach any given second bounding circle, save in the case of the circle $\zeta = -\zeta_0$, which is symmetrical to the original bounding circle with respect to the point of support. This can be made the second bounding circle by giving β the value in (72.5). As β is made indefinitely large the second bounding circle approaches this circle $\zeta = -\zeta_0$ asymptotically.

73. The uniform precessional motion of the top. A comparison of (69.4) and (71.1) shows that the top can execute a forced uniform precession under the applied moment furnished by its own weight, provided that

$$Mgl = \mu[C\nu + (C - A)\mu \cos\theta], \tag{73.1}$$

where μ and ν are the values of $\dot{\phi}$ and $\dot{\psi}$ respectively. These values are necessarily constant when θ is constant, as is seen from (71.5) and (71.6). The question as to what value must be given to β, the value of n being previously assigned, in order that uniform precession may take place with a given angle of precession θ_0 is answered by putting $\zeta = \zeta_0 = \cos\theta_0$ in (72.3); for this implies that the two bounding circles coincide. It is necessary that $b^2n^2 \geqq 2\,a\zeta_0$ in order that the value of β determined by (72.3) should be real. Hence the top may always be made to precess, with any value of n, in a circle below the point of support; but for a circle above the point of support it will not precess uniformly unless the spin n is large enough to make $b^2n^2 \geqq 2\,a\zeta_0$, that is,

$$C^2n^2 \geqq 4\,AMgl\cos\theta_0. \tag{73.2}$$

The values of β necessary for the steady precession are, by (72.3),

$$\beta = \frac{bn(1 + \zeta_0^2) \pm (1 - \zeta_0^2)(b^2n^2 - 2\,a\zeta_0)^{\frac{1}{2}}}{2\,\zeta_0}, \tag{73.3}$$

the corresponding values of $\dot{\phi}$ and $\dot{\psi}$ being

$$\dot{\phi} = \frac{\beta - bn\zeta_0}{1 - \zeta_0^2} = \frac{bn \pm (b^2n^2 - 2\,a\zeta_0)^{\frac{1}{2}}}{2\,\zeta_0}, \tag{73.4}$$

$$\dot{\psi} = n - \dot{\phi}\zeta_0 = n\left(1 - \frac{b}{2}\right) \mp \frac{1}{2}(b^2n^2 - 2\,a\zeta_0)^{\frac{1}{2}}. \tag{73.5}$$

To each value of θ_0 there are, accordingly, in general, two precessions, and these are referred to as the fast and slow precessions. If the conditions for a steady precession are not exactly satisfied, the top will have a slight nutational motion. But the two bounding circles will be very close together when the divergence from the conditions necessary for a steady precession is small. In other words, the uniform precessional motion of the top may be called a *stable state of steady motion.*

74. Discussion of the top which can become vertical. We have seen that if the axis of figure can come into coincidence with the upward vertical, we must have $G_\zeta = Cn$, that is, $\beta = bn$; whereas, if the axis of figure can come into coincidence with the downward vertical, we must have $\beta = -bn$. The two cases may best be discussed separately.

CASE I. $\beta = bn$.

Here $\zeta = 1$ is a zero of the cubic polynomial $f(\zeta)$ of (71.7), and the other two zeros are furnished by the quadratic equation

$$\frac{f(\zeta)}{1-\zeta} = 0, \quad \text{or} \quad (\alpha - a\zeta)(1+\zeta) - b^2n^2(1-\zeta) = 0. \tag{74.1}$$

If ζ_0 is the initial value of ζ the quadratic polynomial $\dfrac{f(\zeta)}{1-\zeta}$ must be positive or zero for $\zeta = \zeta_0$; for neither $f(\zeta_0) = \left(\dfrac{d\zeta}{dt}\right)_0^2$ nor $(1-\zeta_0)$ can be negative. For $\zeta = -1$, the quadratic has the negative value $-2b^2n^2$, so that it has a zero in the interval $\zeta_0 \geqq \zeta > -1$. For $\zeta = 1$, the quadratic has the value $2(\alpha - a)$, whereas for large values of ζ it is negative. Hence the third zero of $f(\zeta)$ is greater than 1 if $\alpha > a$; whereas if $\alpha < a$, it lies between 1 and ζ_0. If $\alpha = a$, the cubic $f(\zeta)$ has $\zeta = 1$ as a double zero. It follows, then, that if $\alpha < a$, the vertex of the top lies, as before, between two bounding circles, and the axis of figure never becomes vertical. If $\alpha > a$, the upper bounding circle reduces to the highest point $\zeta = 1$ of the sphere. At this point the energy integral (71.3 a) gives $\dot\theta^2 = \alpha - a$, so that the vertex of the top passes through the highest point of the sphere, descending on the other side of the unit sphere. The angular velocities $\dot\phi$ and $\dot\psi$ are, from (71.5 a) and (71.6 a), given by

$$\dot\phi = \frac{bn}{1+\zeta}, \quad \dot\psi = n - \frac{bn\zeta}{1+\zeta}, \tag{74.2}$$

so that $\dot\phi$ always keeps the same sign.

If $\alpha = a$, the polynomial $f(\zeta)$ has the double zero $\zeta = 1$, the third zero being $\frac{b^2n^2}{a} - 1$. If this is greater than or equal to unity, the axis of figure must have been originally vertical, and it remains so. This is a limiting case of the uniform stable precessions discussed in the previous section, and it gives an explanation of the "sleeping top." The condition $b^2n^2 \geqq 2\,a$ is equivalent to

$$C^2n^2 \geqq 4\,AMgl. \tag{74.3}$$

If this condition is not satisfied, there is a lower bounding circle, the highest point on the sphere playing the rôle of the upper bounding circle. As $f(\zeta)$ has here a double zero, the integration of the equation (71.7) does not involve the elliptic functions in this case. Writing $\zeta_3 = \frac{b^2n^2}{a} - 1$, we find

$$\frac{\zeta - \zeta_3}{1 - \zeta_3} = \tanh^2 \frac{[a(1 - \zeta_3)]^{\frac{1}{2}}}{2}\, t, \tag{74.4}$$

so that the axis of figure would take an infinite time to reach the upward vertical.

Case II. $\beta = -\,bn$.

Here $\zeta = -1$ is a zero of $f(\zeta)$, the other two zeros being those of the quadratic polynomial

$$\frac{f(\zeta)}{1 + \zeta}, \quad \text{or} \quad (\alpha - a\zeta)(1 - \zeta) - b^2n^2(1 + \zeta). \tag{74.5}$$

If ζ_0 is the initial value of ζ, this quadratic has a zero between ζ_0 and 1, and a zero greater than unity. The lowest point of the unit sphere plays the rôle of the lower bounding circle. At this lowest point our energy integral gives $\dot{\theta}^2 = \alpha + a$, and, if this is zero, a second zero of $f(\zeta)$ coincides with the original one $\zeta = -1$. In this case the top must originally have had its axis of figure vertically downward, and it stays thus permanently, no matter how large n is made.

75. The original Foucault gyroscope. The usual arrangement for supporting a gyroscope is known as a Cardan bracket. It consists of an outer ring which can turn about a fixed diameter $O\zeta$, thus contributing the component $\dot{\phi}$ of the angular velocity $\boldsymbol{\omega}$ of the gyroscope. This outer ring supports an inner concentric ring which can also turn about a diameter ON, the line of nodes, perpendicular to $O\zeta$. The rotation of the inner ring contributes the component $\dot{\theta}$ of $\boldsymbol{\omega}$. The gyroscope itself is pivoted to the inner ring, the pivots being at the end of the diameter which is perpendicular to the line of nodes. The spin $\dot{\psi}$ of the gyroscope about its axis of figure gives the remaining component of $\boldsymbol{\omega}$. The common center O of the two rings plays the rôle of the fixed point of support.

Let us now examine what effect the rotation of the earth has upon the motion of such a gyroscope. As has been already pointed out in § 51, we may disregard the acceleration of the center of the earth with respect to an inertial frame. The point of support will not, however, be at rest in a frame whose origin is at the center of the earth and which has no rotation with respect to an inertial frame, but will have an acceleration due to the earth's rotation. If, nevertheless, the gyroscope is balanced, we may disregard the motion of the point of support and consider merely the motion of the gyroscope relative to its center of mass (§ 52). Since the gravitational force of the earth on the gyroscope is equivalent to a single force through its center of mass, the motion about the center of mass will take place under no moments. If, then, the angular-velocity vector $\boldsymbol{\omega}$ is originally directed along the axis of figure, — that is, if $\dot{\theta}$ and $\dot{\phi}$ are originally zero, — it will continue to be directed along the axis of figure, which will be a fixed line in space. The motion of the gyroscope about its center of mass is merely a rotation with constant velocity about a permanent axis through the center of mass. Now any framework of reference whatever which is attached to the earth has, relatively to an inertial frame, an angular velocity of magnitude $\omega = \frac{2\pi}{86{,}164}$ radians per second about the polar axis of the earth. Choosing a reference frame with origin at O, the point of support, this angular velocity may be resolved into two parts, one of magnitude $\omega \cos \theta = \omega \sin L$ about the upward vertical, and the other of magnitude $\omega \sin \theta = \omega \cos L$ about the northward horizontal line. Here L is the latitude of O, regarded as positive in the Northern Hemisphere and as negative in the Southern Hemisphere. The axis of figure, which has a fixed direction in an inertial frame, will, accordingly, when viewed from the earth, have an angular velocity with the two components (1) $\omega \sin L$ about the downward vertical, (2) $\omega \cos L$ about the horizontal line toward the south.

If we suppose the Cardan bracket so arranged that $O\zeta$ is the upward vertical, and if we start the gyroscope spinning about the axis of figure placed horizontally (to minimize friction on the pointed ends of the axis of figure at the pivots on the inner ring), the axis of figure will remain practically horizontal for several minutes, since ω is so small. The outer ring should then move from north to east in the Northern Hemisphere, and from north to west in the Southern Hemisphere, its angular velocity $\dot{\phi}$ being the vertical component $\omega \sin L$ of the angular velocity of an inertial frame with respect to the earth (both the line of nodes and the axis of figure being in the horizontal plane through O). For a duration of about eight minutes the angle through which the outer ring should move, at the latitude of Paris, amounts to about one and one-half degrees, — a fact which can be easily observed with the aid of a microscope. It was from this experiment, arranged by Foucault in 1852 to demonstrate the rotation of the earth, that the gyroscope takes its name. The angular velocity ω is, however, so small that only its direction — not a correct quantitative calculation of its magnitude — could be obtained by this arrangement. A very slight displacement of the center of mass from the point of support O introduces a motion of the axis of figure, due to the moment about O of the weight of the gyroscope, which is comparable to the motion due to the rotation of the earth. An arrangement in which the center of mass is deliberately displaced from the center of support, by means of a weight attached to the axis of figure, is better suited to the determination of ω; and this arrangement, known as the barygyroscope, will be discussed in detail in the next section.

76. The gyroscope with two degrees of freedom; determination of the angular velocity of the earth; the gyroscopic compass; gyroscopic determination of latitude. When a dynamically symmetrical body, spinning with uniform angular velocity $\dot{\psi}$ about its axis of figure, is forced to move in a steady precession with uniform angular velocity $\dot{\phi} = \mu$ about a fixed line, it experiences a deviation moment

$$\mathbf{D} = \left[C + (C - A)\frac{\mu}{\nu}\cos\theta\right][\boldsymbol{\nu}.\boldsymbol{\mu}],$$

which tends to set the axis of figure along the axis of precession (§ 69). If, then, we suppose the axis of figure of our gyroscope constrained by some arrangement to move in a plane fixed with respect to the earth, we shall have a case of this forced precession, the axis of precession being a parallel through O to the polar axis of the earth, and $\mu = \frac{2\pi}{86{,}164}$ being the angular velocity of the earth. Here μ is so small that we may neglect the ellipsoidal part $(C - A)\frac{\mu}{\nu}\cos\theta\,[\boldsymbol{\nu}.\boldsymbol{\mu}]$ of the deviation moment in comparison with the spherical part $C[\boldsymbol{\nu}.\boldsymbol{\mu}]$. If the necessary applied moment $\mathbf{M} = -\mathbf{D} = C[\boldsymbol{\mu}.\boldsymbol{\nu}]$ is not supplied, the axis of figure will not execute the forced precession which it would have if the applied moment were present, and it will move in the plane in which it is constrained. If we analyze $\boldsymbol{\mu}$ into two parts the first of which, $\boldsymbol{\mu}_1$, is in the plane in which the axis of figure is constrained to move, the second, $\boldsymbol{\mu}_2$, being perpendicular to this plane, there will be a corresponding analysis of the deviation moment $\mathbf{D}$ into two parts, $\mathbf{D}_1 = C[\boldsymbol{\nu}.\boldsymbol{\mu}_1]$ and $\mathbf{D}_2 = C[\boldsymbol{\nu}.\boldsymbol{\mu}_2]$. The part $\mathbf{D}_2$ is in the plane of the axis of figure, and so its effect on the axis of figure is to tend to rotate the axis of figure out of the plane in which it is forced to lie. In this case $\mathbf{D}_2$ will be neutralized, or, in other words, the corresponding part $\mathbf{M}_2$ of the necessary applied moment will be supplied by the reaction moment of the supports which constrain the axis of figure to stay in the plane fixed relatively to the earth. The effective part, as far as the motion of the axis of figure in its plane is concerned, is therefore

$$\mathbf{D}_1 = C[\boldsymbol{\nu}.\boldsymbol{\mu}_1]. \tag{76.1}$$

This is perpendicular to the plane of motion of the axis of figure and has the magnitude $C\nu\mu_1 \sin\epsilon$, where ϵ is the angle from the vector $\boldsymbol{\mu}_1$ to the axis of figure. The effect of $\mathbf{D}_1$ is to make the axis of figure move toward $\boldsymbol{\mu}_1$, so that the motion of the axis of figure will be an oscillatory one about $\boldsymbol{\mu}_1$ as a mean position (provided no other moments are acting on the gyroscope).

On account of the difficulty experienced in suspending the gyroscope so that its center of mass is exactly at the point of support, let us suppose that the effect of gravity is equivalent to a force $\mathbf{P}$ acting at a point on the axis of figure whose distance from the point of support O is l, l being measured positively in that direction along the axis of figure about which the spin $\dot{\psi}$

is positive. The moment of **P** about O may be resolved into two parts, one of which is perpendicular to the plane of the axis of figure, while the other lies in this plane. As before, we see that the latter component is ineffective as far as motion of the axis of figure is concerned. Let us denote the component of **P** in the plane of the axis of figure by $\mathbf{P}_1$, and let the angle from $\boldsymbol{\mu}_1$ to $\mathbf{P}_1$, measured positively in the same sense as ϵ, be α. Then the part of the moment of **P** which tends to increase ϵ is $P_1 l \sin(\alpha - \epsilon)$. The total moment, both deviational and applied, tending to increase ϵ is therefore

$$-C\nu\mu_1 \sin\epsilon - P_1 l \sin(\epsilon - \alpha). \tag{76.2}$$

This may be written as a single harmonic term in the form

$$-R\sin(\epsilon - \epsilon_0), \tag{76.2a}$$

where $$R^2 = (C\nu\mu_1 + P_1 l\cos\alpha)^2 + (P_1 l \sin\alpha)^2,$$

and $$\tan\epsilon_0 = \frac{P_1 l \sin\alpha}{C\nu\mu_1 + P_1 l\cos\alpha}, \tag{76.3}$$

the angle ϵ_0 being determined uniquely by the fact that

$$R\sin\epsilon_0 = P_1 l\sin\alpha \quad \text{and} \quad R\cos\epsilon_0 = C\nu\mu_1 + P_1 l\cos\alpha.$$

Since the moment of inertia of the gyroscope about any axis through O perpendicular to the axis of figure is A, we have, from (69.8),

$$A\ddot{\epsilon} = -R\sin(\epsilon - \epsilon_0), \tag{76.4}$$

so that the axis of figure oscillates about the equilibrium position $\epsilon = \epsilon_0$ like a simple pendulum of length $\frac{Ag}{R}$. For small oscillations the period is $T = 2\pi\left(\frac{A}{R}\right)^{\frac{1}{2}}$.

We may now discuss in some detail the two most important cases, which occur when the plane in which the axis of figure is constrained to lie is (1) a vertical plane or (2) a horizontal plane.

CASE I. *Plane of constraint vertical.* Here $P_1 = P$. Let us introduce for the moment a reference frame whose x-axis is along $\boldsymbol{\mu}_1$ and whose y-axis is in the plane of the axis of figure, the rotation from the x-axis to the y-axis being in the direction in which ϵ is measured positively. The components in this frame of the unit vector along the upward vertical are $(-\cos\alpha, -\sin\alpha, 0)$. Denoting by β the angle between $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_1$, which is acute by the definition of $\boldsymbol{\mu}_1$, the components of the unit vector along

$\boldsymbol{\mu}$ are $(\cos\beta, 0, \sin\beta)$. On taking the scalar product of these two unit vectors and observing that the angle between them is the colatitude, we obtain

$$\sin L = -\cos\alpha\cos\beta. \tag{76.5}$$

If we take the vector product of the unit vector along $\boldsymbol{\mu}$ and the unit vector along the upward vertical, we obtain a vector whose components are

$$(\sin\alpha\sin\beta,\ -\cos\alpha\sin\beta,\ -\sin\alpha\cos\beta). \tag{76.6}$$

This vector has the magnitude $\cos L$ and is directed due east in the horizontal plane. The z-axis also lies in the horizontal plane, and the angle from the vector (76.6) to the z-axis is the angle between the vertical plane toward the north and the vertical plane containing the axis of figure. This angle we shall denote by δ, so that δ is the azimuth of the plane of the axis of figure measured from the north line. We have, from (76.6),

$$\cos L\cos\delta = -\sin\alpha\cos\beta. \tag{76.7}$$

We may use the results (76.5) and (76.7) to replace the angles α and β by the angles L and δ. Writing $\mu_1 = \mu\cos\beta$ and $P_1 = P$ in (76.3), we find

$$\tan\epsilon_0 = \frac{Pl\cos L\cos\delta}{Pl\sin L - C\nu\mu\cos^2\beta}. \tag{76.8}$$

The formula for the deviation of the position of equilibrium from the vertical is somewhat simpler. The deviation from the downward vertical being $\epsilon_0 - \alpha$, we have

$$\tan(\epsilon_0 - \alpha) = \frac{\tan\epsilon_0 - \tan\alpha}{1 + \tan\epsilon_0\tan\alpha} = \frac{-C\nu\mu_1\sin\alpha}{P_1 l + C\nu\mu_1\cos\alpha}$$

in the general case. In the present case this reduces to

$$\tan(\epsilon_0 - \alpha) = \frac{C\nu\mu\cos L\cos\delta}{Pl - C\nu\mu\sin L}. \tag{76.9}$$

When $\nu = 0$, this is zero, which says merely that the body hangs vertically downward. When the spin of the gyroscope is not zero, the deviation of the mean position of its oscillatory motion in its vertical plane is zero only at the poles, $L = \pm\frac{\pi}{2}$, or when the vertical plane is an east-west plane, $\delta = \pm\frac{\pi}{2}$. In a given latitude the deviation of the mean position from the downward vertical is greatest when $\delta = 0$, that is, when the vertical plane points north and south. For a given azimuth the deviation increases as we move toward the equator.

Foucault tried to demonstrate the rotation of the earth by means of a gyroscope whose axis was constrained to move in a vertical plane. In his experiment, however, the gyroscope was, as nearly as possible, balanced; that is, l was as nearly as possible zero. The deviation of the mean position should then have been $\frac{\pi}{2} - L$ when the vertical plane was a north-and-south plane. In this case also $\beta = 0$ and $R = C\nu\mu$, so that the period of the oscillations should be $2\pi\left(\frac{A}{C\nu\mu}\right)^{\frac{1}{2}}$. This arrangement was therefore calculated to determine not only the rotation μ of the earth but the latitude of the position on the earth at which the experiment was being conducted. However, on account of the difficulty of getting the gyroscope exactly balanced, quantitatively correct results are not possible with this arrangement. Gilbert overcame the difficulty by attaching a weight to the axis of figure, the apparatus being then known as the barygyroscope, to indicate the rôle played by gravity. The deviation from the downward vertical is given by (76.9); and we notice that in the Northern Hemisphere, where L is positive, the deviation and the period of the vibrations are greater, for equal numeric values of l, when l is positive than when l is negative. The reverse holds true in the Southern Hemisphere. By making Pl nearly equal to $C\nu\mu \sin L$, the deviation may be made nearly equal to a right angle. The reader wishing more details may consult a paper by A. Denizot, in Jahresberichte des deutschen mathematischen Vereins (1914), 23, p. 445.

CASE II. *Plane of constraint horizontal.* When the plane of the axis of figure is horizontal, $P_1 = 0$, and gravity has no effect on the motion of the axis of figure in the horizontal plane in which it is constrained to move. The angle β, being the complement of the angle made by the perpendicular to the plane of constraint with μ, is here the angle between the upward vertical and μ, that is, the latitude L. The equilibrium position, by (76.8), is given by $\epsilon_0 = 0$. Hence the mean position of the vibrating gyroscope is the north-and-south line. Since

$$R = C\nu\mu_1 = C\nu\mu \cos L,$$

the arrangement serves to determine the meridian and $\mu \cos L$. The instrument is known as the gyroscopic compass, and it serves also to determine the latitude. In the actual setting up of the instrument it is necessary to avoid friction as much as possible; and so, following a suggestion of Lord Kelvin, the apparatus is usually arranged to float on mercury. Knowing L, μ can be calculated by this instrument with an error of about 2 per cent. A good account is given in an article by H. Lorenz, "Technische Anwendungen der Kreiselbewegung," in *Zeitschrift des Vereins deutscher Ingenieure* (1919), **9**, p. 1224.

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See bibliography for Chapter IX, pp. 228–229.

EXERCISES

1. Prove that if a ship is in motion with a speed whose northerly component is v, the gyrostatic compass will (when in equilibrium) point to the west of true north by an amount $\dfrac{v}{\omega a \cos \lambda}$, where a is the radius of the earth.

2. A uniform circular disk has its center O fixed, and moves under the action of no external forces. The disk is given an initial angular velocity ω about a diameter coinciding with the fixed X-axis OX, and a velocity n about its own axis, which then coincides with the fixed Z-axis OZ. Show that, at any subsequent time t,

$$\theta = 2 \operatorname{arc\,sin}\left(\frac{\omega}{\alpha} \sin \frac{\alpha t}{2}\right), \qquad (\alpha = (\omega^2 + 4\,n^2)^{\frac{1}{2}})$$
$$\phi = \operatorname{arc\,cot}\left(\frac{\omega}{\alpha} \tan \frac{\alpha t}{2}\right),$$

where θ is the angle between the axis Oz of the disk and OZ, and ϕ is the angle between the plane ZOX and the plane ZOz.

3. A body can rotate freely about a fixed vertical axis for which its moment of inertia is I. The body carries a second body in the form of a disk which can rotate about a horizontal axis, fixed in the first body and intersecting the vertical axis. In the position of equilibrium the moments and products of inertia of the disk with respect to the vertical and horizontal axes are, respectively, A, B, F. Prove that if the system starts from rest with the plane of the disk inclined at an angle α to the vertical, the first body will oscillate through an angle $\dfrac{2\,F}{[B(A+I)]^{\frac{1}{2}}} \operatorname{arc\,tan}\left[\dfrac{B^{\frac{1}{2}} \sin \alpha}{(A+I)^{\frac{1}{2}}}\right]$.

CHAPTER XI

GENERAL DYNAMICAL THEOREMS

77. Generalized coördinates. We shall first consider a system of k material particles whose coördinates relative to any convenient Cartesian reference frame are (x_s, y_s, z_s) respectively, where $s = 1, 2, \cdots, k$. The usual situation will be that the positions of these various particles are not independent of one another but are interconnected in such a way that the $3\,k$ coördinates (x_s, y_s, z_s) satisfy a certain number r of independent equations

$$f_q(x_1, y_1, z_1, \cdots, x_k, y_k, z_k) = 0. \quad (q = 1, 2, \cdots, r) \qquad (77.1)$$

It may even happen that these equations involve the time t explicitly, in addition to the coördinates of some or all of the points of the system. The equations (77.1) are said to define the constraints imposed upon the system of material particles; and when these equations involve the time explicitly, we speak of moving constraints. For example, if we are considering only one material particle which is constrained to move upon a given surface, the equation of this surface will define the constraint; and if the given surface is a moving one, the time will enter its equation as a parameter. The r independent equations (77.1) defining the constraints enable us to express r of the coördinates of the k particles in terms of the remaining $3\,k - r$; but it is more symmetrical and convenient to suppose that each of the $3\,k$ coördinates is expressed in terms of $3\,k - r = m$ independent variables $(q_1, q_2, \cdots, q_m)$. If the constraints are moving, the time will occur explicitly in the expressions giving the position of each particle in terms of the variables q. These variables are known as generalized coördinates, it being clearly understood that they need not have the dimension of a length. The position vector of each particle is given in terms of the generalized coördinates by an equation of the type

$$\mathbf{r}_s = \mathbf{r}_s(q_1, q_2, \cdots, q_m, t). \quad (s = 1, 2, \cdots, k) \qquad (77.2)$$

On differentiation with respect to the time, we derive from these the velocities of the various particles of the system

$$\dot{\mathbf{r}}_s = \frac{\partial \mathbf{r}_s}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}_s}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \mathbf{r}_s}{\partial q_m}\dot{q}_m + \frac{\partial \mathbf{r}_s}{\partial t}. \tag{77.3}$$

These serve to define the kinematic displacement differentials

$$d\mathbf{r}_s = \frac{\partial \mathbf{r}_s}{\partial q_1}dq_1 + \cdots + \frac{\partial \mathbf{r}_s}{\partial q_m}dq_m + \frac{\partial \mathbf{r}_s}{\partial t}dt. \tag{77.4}$$

In addition to these differentials we shall find it necessary to consider what they would reduce to if the time did not enter the equations of constraint. Introducing the notation $\delta\mathbf{r}_s$, we have the equations of definition

$$\delta\mathbf{r}_s = \frac{\partial \mathbf{r}_s}{\partial q_1}dq_1 + \cdots + \frac{\partial \mathbf{r}_s}{\partial q_m}dq_m. \tag{77.5}$$

These differentials are said to define a virtual displacement of the various particles of the system, but it must be clearly understood that they have a mathematical rather than a physical significance. To illustrate the difference between the kinematic displacement differentials and the virtual, or geometric, differentials, let us consider the case of a single particle constrained to stay on a moving surface. Here we have two generalized coördinates, q_1 and q_2. The kinematic differential $d\mathbf{r}$ gives the actual direction of motion of the particle at any instant, and the geometric differential $\delta\mathbf{r}$ gives a direction tangent to the position of the moving surface at the instant in question. Since the surface itself is supposed to be moving, the direction of motion of the particle is not, in general, tangent to the surface, — a fact which shows the difference between the two differentials.

If we consider all possible motions of the system through a given position, instead of confining our attention to the actual motion taking place, the dq's occurring in (77.5) will be arbitrary numbers instead of being proportional to the $\dot{q}$'s of (77.3). In order to avoid a quite possible confusion of meaning, we shall use the symbol δq when we wish to refer to all possible motions, so that the δq's are arbitrary numbers. When the constraints are not moving, so that the time variable does not enter the equations (77.2) explicitly, the infinity of virtual displacements (found by giving arbitrary numeric values to the δq) includes one displacement which is identical with the actual kinematic displacement differential. This is the one for which each δq is put equal to the corresponding dq.

Rewriting (77.5) in the form

$$\delta \mathbf{r}_s = \frac{\partial \mathbf{r}_s}{\partial q_1}\,\delta q_1 + \cdots + \frac{\partial \mathbf{r}_s}{\partial q_m}\,\delta q_m, \qquad (77.5\,a)$$

we form the scalar product of the relative force $\mathbf{F}_s$ acting on the sth particle of the system and the virtual displacement $\delta \mathbf{r}_s$ of this particle, and then add the results obtained in this way for all the particles of the system. The number

$$\Sigma_s(\mathbf{F}_s . \delta \mathbf{r}_s) \qquad (77.6)$$

so found is called the *virtual work* of the forces acting on the system of particles. It follows from (77.5 a) that the virtual work of the forces acting on the system may be written in the form

$$Q_1\,\delta q_1 + Q_2\delta q_2 + \cdots + Q_m\,\delta q_m, \qquad (77.7)$$

where

$$Q_k = \Sigma_s\left(\mathbf{F}_s \cdot \frac{\partial \mathbf{r}_s}{\partial q_k}\right). \quad (k = 1, 2, \cdots, m) \qquad (77.8)$$

The m quantities Q_k are called generalized-force components; but it is clear that they will not have the dimensions of a force unless the corresponding coördinate q_k has the dimensions of a length. It is the product $Q_k\delta q_k$ which has a direct physical interpretation, and this product has the dimensions of work or energy.

Since the δq's in (77.7) are arbitrary numbers, it is clear that the necessary and sufficient condition for the virtual work of an applied-force system to be zero is that all the generalized-force components Q be zero. Now the advantage of the concept of virtual work is that the motion of the dynamic system can be determined without knowing the force acting on each individual particle of the system. All that it is necessary to know is the virtual work of the force system, which is equivalent to knowing the generalized-force components. It is evident from the expression (77.6) that those forces that are perpendicular to the virtual displacements of the particles on which they act will not contribute anything to the expression for the virtual work. For example, in the case of a particle constrained to stay on a surface (moving or not) a force normal to the surface will not affect the expression for the virtual work. Therefore if the reaction of the constraining surface on the particle is perpendicular to the surface, it is unnecessary to know this reaction in order to write down the virtual work or to find the generalized-force components Q. In this case the constraining surface is said to be smooth or frictionless. Frequently a dynamic system is such that the virtual work of the constraints is so small in comparison with the virtual work of the other forces involved that we may neglect it, and it is, accordingly, possible to determine the generalized-force components without knowing the reactions of the constraints. The motion of such a dynamic system can be determined if the external applied forces are given, and the dynamic system is said to be frictionless.

78. Lagrange's equations. Denoting the mass and acceleration of the sth particle by m_s and $\mathbf{A}_s$ respectively, we have $\mathbf{F}_s = m_s\mathbf{A}_s$. The virtual work of the forces acting on the system may be written, then, in the form

$$\Sigma_s(m_s\mathbf{A}_s . \delta\mathbf{r}_s).$$

We have, accordingly, from (77.7), the equation

$$\Sigma_s(m_s\mathbf{A}_s . \delta\mathbf{r}_s) = Q_1\,\delta q_1 + Q_2\,\delta q_2 + \cdots + Q_m\,\delta q_m. \qquad (78.1)$$

The left-hand side of this equation may be transformed exactly as in § 43. From the equations

$$\mathbf{r}_s = \mathbf{r}_s(q_1, q_2, \cdots, q_m, t),$$

we obtain

$$\delta\mathbf{r}_s = \frac{\partial\mathbf{r}_s}{\partial q_1}\,\delta q_1 + \cdots + \frac{\partial\mathbf{r}_s}{\partial q_m}\,\delta q_m.$$

Hence the coefficient of any of the δq's (δq_k, let us say) on the left-hand side of (78.1) is $\Sigma_s\left(m_s\ddot{\mathbf{r}}_s \cdot \frac{\partial\mathbf{r}_s}{\partial q_k}\right)$. By following the method of § 43, this is seen to be P_k, where

$$P_k = \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \qquad (78.2)$$

and

$$T = \tfrac{1}{2}\,\Sigma_s m_s\dot{\mathbf{r}}_s{}^2$$

is the kinetic energy of the system. We may therefore rewrite equation (78.1) in the form

$$P_1\,\delta q_1 + P_2\,\delta q_2 + \cdots + P_m\,\delta q_m = Q_1\,\delta q_1 + Q_2\,\delta q_2 + \cdots + Q_m\,\delta q_m.$$

Since the δq's are arbitrary numbers, this implies that

$$P_s = Q_s. \quad (s = 1, 2, \cdots, m)$$

Writing for P_s its expression in (78.2), we have the m equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_s}\right) - \frac{\partial T}{\partial q_s} = Q_s. \quad (s = 1, 2, \cdots, m) \qquad (78.3)$$

These are second-order differential equations for the m generalized coördinates q, the independent variable being the time. They are known as the Lagrangian equations governing the motion of the system. In a discussion of the motion of the dynamic system these equations have the advantage over a direct discussion of the motion of each individual particle of the system, in that it is unnecessary to know the reactions of the frictionless constraints. If these reactions are desired, there is usually no particular advantage gained by using the Lagrangian equations. In order to compare the two methods, the student may discuss the motion of a particle on a smooth sphere under the action of gravity.

It may be remarked that the generalized coördinates were assumed independent only at the end of the argument, where we wished to make use of the arbitrariness of the δq's. It is very often convenient to use a superfluous number of generalized coördinates connected with each other by one or more relations of the type

$$f(q_1, q_2, \cdots, q_r, t) = 0. \tag{78.4}$$

Let us suppose, for instance, that there are p of these relations, so that there are in reality only $r - p = m$ independent generalized coördinates q. However, it frequently happens that the solution of the p relations for p of the interdependent coördinates q in terms of the remaining m is either difficult or inconvenient, and we wish to know how to proceed without having recourse to this solution. The δq's are no longer arbitrary numbers but are connected with one another by p relations, found by differentiating the p relations (78.4). The time variable plays the rôle of a constant in these differentiations. These relations are

$$\frac{\partial f_k}{\partial q_1}\delta q_1 + \frac{\partial f_k}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f_k}{\partial q_r}\delta q_r = 0. \quad (k = 1, 2, \cdots, p) \tag{78.5}$$

Upon multiplying each of these p equations by an arbitrary multiplier λ_k and subtracting the results from the equation

$$(P_1 - Q_1)\delta q_1 + (P_2 - Q_2)\delta q_2 + \cdots + (P_r - Q_r)\delta q_r = 0,$$

which we have already obtained, we may avail ourselves of the arbitrariness of the multipliers λ_k to make the coefficients of the differentials δq of the dependent coördinates zero, whereupon the remaining coefficients of the differentials of the independent coördinates will have to be zero as before. We derive in this way r equations,

$$P_k = Q_k + \lambda_1 \frac{\partial f_1}{\partial q_k} + \cdots + \lambda_p \frac{\partial f_p}{\partial q_k}, \quad (k = 1, 2, \cdots, r) \tag{78.6}$$

which take the place of the previous equations $P_k = Q_k$. There are now $r + p$ unknowns, as, in addition to the r coördinates q, we have the p unknown multipliers λ. To determine these $r + p$ unknowns we have the r equations (78.6) and the p equations (78.4). If we wish to give a physical interpretation to the mathematical procedure explained here, we say that the equations (78.4) imply constraints imposed on the dynamic system, and that the terms $\lambda_s \frac{\partial f_s}{\partial q_k}$ occurring in (78.6) are the generalized components of the forces of reaction due to these constraints.

As an illustration of the use of a superfluous number of coördinates, let us consider the simple problem of the motion of two masses m_1 and m_2 connected by a light inextensible string over a smooth pulley, as in Atwood's machine. If z_1 and z_2 denote the vertical portions of the string, they are connected by a single relation

$$z_1 + z_2 - l = 0.$$

The virtual work is $m_1 g \delta z_1 + m_2 g \delta z_2$, so that $Q_1 = m_1 g$ and $Q_2 = m_2 g$. The kinetic energy T of the system of two particles is $\frac{1}{2} m_1 \dot{z}_1{}^2 + \frac{1}{2} m_2 \dot{z}_2{}^2$, so that $P_1 = m_1 \ddot{z}_1$ and $P_2 = m_2 \ddot{z}_2$. The two equations (78.6) are $m_1 \ddot{z}_1 = m_1 g + \lambda$, $m_2 \ddot{z}_2 = m_2 g + \lambda$. From the relation of the coördinates z_1 and z_2, we obtain, on differentiating twice with respect to the time, the equation $\ddot{z}_1 + \ddot{z}_2 = 0$. On eliminating the multiplier λ, we find $\ddot{z}_1 = -\ddot{z}_2 = \frac{(m_1 - m_2)g}{m_1 + m_2}$. This is the theory of Atwood's machine which is used for the determination of g. The multiplier λ gives the negative tension of the string. We find at once that

$$\lambda = \frac{-2\, m_1 m_2 g}{m_1 + m_2}.$$

79. General remarks on Lagrange's equations. From the equations (77.3) it is at once seen that the kinetic energy T of our dynamic system is a quadratic function of the coördinate velocities $\dot{q}_s$. If the time does not enter the equations (77.2) explicitly, so that the velocity $\dot{\mathbf{r}}_s$ of any particle of the system is a linear homogeneous function of the coördinate velocities $\dot{q}$, the kinetic energy will be a homogeneous quadratic function of these coördinate velocities. This is the simplest and most important case.

In the general case, however, the expression for T will involve terms linear in the coördinate velocities $\dot{q}$, and also terms independent of these coördinate velocities. We shall denote the quadratic terms of T by T_2, the linear terms by T_1, and the terms independent of the coördinate velocities by T_0; so that

$T = T_2 + T_1 + T_0$. In the usual case where the time does not enter explicitly the equations of connection, this expression for T will reduce to the single term T_2. It is well to remark that T_2 is a positively definite form; that is, it can never be negative, and can be zero only when all the coördinate velocities are separately zero. The fact that T_2 can never be negative follows from its definition as $T_2 = \Sigma \frac{1}{2} m \left(\frac{\delta \mathbf{r}}{dt}\right)^2$, where $\frac{\delta \mathbf{r}_s}{dt} = \dot{\mathbf{r}}_s - \frac{\partial \mathbf{r}_s}{\partial t}$, from which it is obvious that T_2 is a sum of square terms none of which can be negative. The only way that T_2 could be zero would be for each of the expressions $\frac{\delta \mathbf{r}_s}{dt}$ to be zero simultaneously. If this could happen, without all the δq's being zero at the same time, the m generalized coördinates q would not be distinct, because we could change their values slightly without changing the position of any particle of the material system. The significance of this remark is that we are assured that the m linear functions of the coördinate velocities $\dot{q}$, obtained by forming the m partial derivatives of the kinetic energy T with respect to these coördinate velocities, are independent functions of the $\dot{q}$'s. Therefore, if we find it convenient, we can solve for the coördinate velocities and express each of them in terms of the m partial derivatives of the kinetic energy with respect to the coördinate velocities. Denoting these partial derivatives by the symbols p, we have the m equations of definition

$$p_s = \frac{\partial T}{\partial \dot{q}_s}. \quad (s = 1, 2, \cdots, m) \tag{79.1}$$

We may solve these m linear equations and express the coördinate velocities in terms of the p's. When this is done, it will be found that the differential equations (78.3) which govern the motion of the dynamic system may be put in a convenient standard form. In the particular case of the motion of a single material particle, moving without constraints, it is at once seen that if the position coördinates q are rectangular Cartesian coördinates, the three quantities p defined by (79.1) are the components of the linear-momentum vector of the particle. In the general case of any dynamic system the name is carried over, and p_s is said to be the generalized-momentum coördinate corresponding to the position coördinate q_s. Since the differential equations

(78.3) are *second*-order differential equations for the m position coördinates q, regarded as functions of the single independent variable t, it is necessary, before the position at any future instant can be predicted, to know not only the position of the system at any given instant but also the coördinate velocities at this instant. The idea of position and velocity of a dynamic system is epitomized in the word *phase*. We may say that if the phase of the system at any given instant is known, the equations (78.3) enable us to predict the phase at any future instant. In order to know the phase of a system, we need to have, in addition to the values of the position coördinates, the values of their time derivatives $\dot{q}$; and we have seen above that this is equivalent to giving, in addition to the values of the q, the values of the generalized momenta p. It is for this reason that the momenta are referred to as "coördinates." In a word, a dynamic system is not sufficiently described by giving its position, but is completely described by its phase at any given instant; and the $2m$ numbers necessary to specify its phase — that is, the q and p — are called the position coördinates and momenta coördinates, respectively, of the system. It will have been noticed that a complete description of the system requires not only a knowledge of the phase but also a knowledge of the particular instant of time at which the system had the phase mentioned; so that, all together, $2m+1$ numbers or coördinates (q, p, t) are necessary for a complete description of the system. These $2m+1$ numbers are referred to as the coördinates of the *state* of the dynamic system, so that the word "state" is equivalent to the hyphenated expression "phase-time." It is found very convenient, in talking about the motion of a dynamic system, to use a kind of geometric language. Thus, if we are interested in the successive phases of the system without being particularly concerned about the times at which these phases are assumed, we could say that the phase of the system is represented by a *point* in a *phase* space of $2m$ dimensions whose coördinates are the (q, p) which give the phase, and that as the dynamic system moves, and so takes different phases, the representative point moves along a curve in the phase space. If we find it necessary to be concerned about the times at which the various phases are assumed, we would use, as representing the state of the dynamic system, a point in a state space of $2m+1$

dimensions whose coördinates are the $2m + 1$ numbers (q, p, t) which give the state of the system. Then as the system moves, or remains at rest, thus occupying different states, the representative point will trace out a curve in the state space, and this curve may be regarded as picturing the life history of the system. The two points of view are identical if we label each point of the representative curve in the phase space with a number giving the time at which the corresponding phase was assumed by the system. In order to prevent possible misconception, it is well to say at this point that when we say we are adopting here a mode of geometric language, the etymological derivation of the word "geometry" is entirely lost sight of. There is no idea of measurement, and we do not attempt to attach any meaning to the phrase "distance between two points" in the representative phase or state space we are talking about. The only reason for introducing the idea of a representative space of a large number of dimensions is that it is convenient to use the abbreviated expression "curve" for the long phrase "successive phases (or states) of the dynamic system" and so on. The various expressions "path," "trajectory of the system," or "integral curve of the dynamic equations (78.3)" are used interchangeably for this idea of the successive phases (or states, as the case may be) of the system. For, as we shall shortly see, it is convenient to consider curves in the representative phase (or state) space which do not picture the phase history (or state history) of the system.

80. The canonical form of the dynamic equations. We may use the equations of definition (79.1) to replace the m *second-order* differential equations (78.3) for the q as functions of the time t by $2m$ differential equations of the *first* order for the q and p as functions of t. These equations are

$$\frac{dq_s}{dt} = f_s(q, p, t), \quad \frac{dp_s}{dt} = \frac{\partial T}{\partial q_s} + Q_s, \quad (s = 1, 2, \cdots, m) \qquad (80.1)$$

where the first set of equations is obtained by solving the equations (79.1) for the $\dot{q}$ as functions of the position coördinates q, the momenta p, and the time t. The equations (80.1) may be given a more symmetrical form if we introduce the function K defined by the equation

$$K = \Sigma_s p_s \dot{q}_s - T, \qquad (80.2)$$

where we suppose the $\dot{q}$ to be expressed in terms of (q, p, t), so that K is a function of these state coördinates. The complete differential of K follows from (80.2), since

$$dK = \Sigma_s(p_s d\dot{q}_s + \dot{q}_s dp_s) - dT\,,$$

and since

$$dT = \Sigma_s\left(\frac{\partial T}{\partial q_s}\,dq_s + \frac{\partial T}{\partial \dot{q}_s}\,d\dot{q}_s\right) + \frac{\partial T}{\partial t}\,dt$$

$$= \Sigma_s\left(\frac{\partial T}{\partial q_s}\,dq_s + p_s d\dot{q}_s\right) + \frac{\partial T}{\partial t}\,dt, \qquad \text{[By (79.1)]}$$

we may write

$$dK = \Sigma_s\left(\dot{q}_s dp_s - \frac{\partial T}{\partial q_s}\,dq_s\right) - \frac{\partial T}{\partial t}\,dt,$$

which is equivalent to the equations

$$\frac{\partial K}{\partial q_s} = -\frac{\partial T}{\partial q_s}, \quad \frac{\partial K}{\partial p_s} = \dot{q}_s, \quad \frac{\partial K}{\partial t} = -\frac{\partial T}{\partial t}. \quad (s = 1, 2, \cdots, m) \quad (80.3)$$

In performing the partial differentiations indicated, K is regarded as a function of (q, p, t), and T is regarded as a function of $(q, \dot{q}, t)$. We may therefore rewrite the dynamic equations (80.1) in the form

$$\frac{dq_s}{dt} = \frac{\partial K}{\partial p_s}, \quad \frac{dp_s}{dt} = -\frac{\partial K}{\partial q_s} + Q_s. \quad (s = 1, 2, \cdots, m) \quad (80.4)$$

This form of the dynamic equations is known as the canonical form; and it may be made even more symmetrical when the generalized forces Q_s are the partial derivatives of a function of the q (and t, possibly), each with respect to its corresponding generalized coördinate q_s. Denoting this function of (q, t) by $-V$, we have, in this case,

$$Q_s = -\frac{\partial V}{\partial q_s}, \quad (s = 1, 2, \cdots, m) \qquad (80.5)$$

and the expression (77.7) for the virtual work of the forces acting on the system takes the form

$$-\left(\frac{\partial V}{\partial q_1}\,\delta q_1 + \cdots + \frac{\partial V}{\partial q_m}\,\delta q_m\right).$$

Defining the function H by the equation

$$H = K + V, \qquad (80.6)$$

so that H is a function of the state coördinates (q, p, t), we have

$$\frac{\partial H}{\partial q_s} = \frac{\partial K}{\partial q_s} - Q_s, \quad \frac{\partial H}{\partial p_s} = \frac{\partial K}{\partial p_s};$$

and our equations (80.4) take the final form

$$\frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}, \quad \frac{dp_s}{dt} = -\frac{\partial H}{\partial q_s}. \quad (s = 1, 2, \cdots, m) \qquad (80.7)$$

In the cases with which we shall be most frequently concerned, the equations of connection (77.2) will not involve the time explicitly, and the kinetic energy T will be a homogeneous quadratic function T_2 of the coördinate velocities $\dot{q}$. It follows, from the equations of definition (79.1), that the expression $\Sigma_s p_s \dot{q}_s = 2\,T$, and so the quantity K defined by (80.2) is equal to T. It is, of course, differently expressed, as K is a function of the phase coördinates (q, p) in this case, whereas T is expressed as a function of the position coördinates q and the coördinate velocities $\dot{q}$. It follows that the function H defined by (80.6) has the value $T + V$. In the particular case we are considering, the kinetic energy T does not involve the time explicitly, nor do the equations (79.1), which enable us to express the coördinate velocities $\dot{q}$ in terms of the momenta p; and so H does not involve the time explicitly, assuming that V does not involve t explicitly. In other words, H is completely determined by the phase of the system, and it is not necessary to know the time at which this phase was assumed. If we calculate the rate of change of H with respect to the ultimate independent variable t as the system moves, we find, on using equations (80.7), that

$$\frac{dH}{dt} = \Sigma_s\left(\frac{\partial H}{\partial q_s}\frac{dq_s}{dt} + \frac{\partial H}{\partial p_s}\frac{dp_s}{dt}\right) = 0. \qquad (80.8)$$

We find, then, that H remains constant despite the varying phase of the system. This constant H, whose expression as a function of the phase coördinates (q, p) dominates the motion of the dynamic system, is called the energy of the system; and the part V which is added to the kinetic energy to yield the total energy H is called the potential energy of the system (or, better, the potential energy of the system in the field of force to which it is subjected). The reader should compare this general result with the particular case of a single moving particle discussed in § 41.

When the equations of connection (77.2) involve the time variable t explicitly, matters are not so simple. The kinetic energy T is then an expression of the form $T_2 + T_1 + T_0$; and it follows, from (79.1), that $\Sigma_s p_s \dot{q}_s = 2\ T_2 + T_1$, the term T_0 not occurring, since it does not involve the coördinate velocities $\dot{q}$. Hence the quantity K, defined by (80.2), has the value $T_2 - T_0$. The quantity H introduced by equation (80.6) has the value $T_2 - T_0 + V$; but since it now involves the time variable t explicitly, it is no longer constant during the motion of the system. On using the equations (80.7), its rate of change is found to be

$$\frac{dH}{dt} = \Sigma_s\left(\frac{\partial H}{\partial q_s}\frac{dq_s}{dt} + \frac{\partial H}{\partial p_s}\frac{dp_s}{dt}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}. \tag{80.9}$$

It may be well to emphasize the fact that even when the equations of connection of the system do not involve the time variable t explicitly, we do not have the energy integral $H =$ constant unless the generalized-force components Q are of the form $Q_s = -\frac{\partial V}{\partial q_s}$, where V is at most a function of $(q_1, \cdots, q_m)$. It sometimes happens that when the components Q are not of this form, they are of the form $Q_s = -\left(\frac{\partial V}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s}\right)$, where F is a homogeneous quadratic function of the coördinate velocities $\dot{q}$, so that, in addition to the term $-\frac{\partial V}{\partial q_s}$, each generalized-force component involves a linear homogeneous combination of the $\dot{q}$'s. We find as before, on availing ourselves of equations (80.7), that

$$\frac{dH}{dt} = -\ \Sigma_s \frac{\partial H}{\partial p_s}\frac{\partial F}{\partial \dot{q}_s} = -\ \Sigma_s \frac{\partial F}{\partial \dot{q}_s}\frac{dq_s}{dt} = -\ 2\ F, \tag{80.10}$$

so that the quadratic expression $2\ F$ measures the rate at which the energy of the system is being *dissipated*. It is known as Rayleigh's dissipation function. The part $-\frac{\partial F}{\partial \dot{q}_s}$ of the generalized-force component Q_s is known as the dissipation part of this force. If we denote it by D_s, and introduce the Lagrangian function

$$L = T - V, \tag{80.11}$$

the Lagrangian equations (78.3) take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = D_s. \tag{80.12}$$

From (80.4) and (80.6) the canonical form of the dynamic equations is readily seen to be

$$\frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}, \quad \frac{dp_s}{dt} = -\frac{\partial H}{\partial q_s} + D_s. \quad (s = 1, 2, \cdots, m) \qquad (80.13)$$

81. Applications of the general theory; cyclic or ignorable coördinates. It often happens that certain coördinates q are such that although their velocities enter the expression for the kinetic energy T, the coördinates themselves do not appear in this expression. For example, in the case of a single moving particle without constraints the three Cartesian coördinates (x, y, z) are of this type; but if we are using space polar coördinates, only the third coördinate ϕ is of this type (see § 43). Examples of coördinates of this type occur when we are discussing the dynamic system consisting of the balls in a ball bearing or of a continuous chain passing over a pulley, and coördinates having this characteristic are called *cyclic coördinates.* For a mathematical reason that will presently appear they are also referred to as *ignorable* coördinates. In the case of cyclic coördinates the Lagrangian equations (78.3) take the simpler form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) = Q_s,$$

or

$$\frac{dp_s}{dt} = Q_s, \qquad (s = 1, 2, \cdots, m) \qquad (81.1)$$

so that the time rate of change of the generalized momentum corresponding to a cyclic coördinate is equal to the corresponding generalized-force component Q_s. This generalized-force component is called the cyclic force; and it follows at once that if any of the cyclic forces either vanishes or is a function of t alone (including, for example, the case where the cyclic force happens to be constant), we can without any trouble integrate the corresponding Lagrangian equation (81.1). When the cyclic force Q_s is zero, the integration yields

$$p_s = \text{constant}. \qquad (81.2)$$

This is known as the momentum integral corresponding to the cyclic coördinate. As an example we may take the case of a particle moving in a plane under the action of a central force which is a function of the distance r from the center of attraction alone. Here $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$, so that θ is a cyclic coördinate; and as $Q_\theta = 0$, we have the angular-momentum integral p_θ (that is, $r^2\dot{\theta}$) $=$ constant (see (49.2)).

When the generalized forces can be written in the form $Q_s = -\frac{\partial V}{\partial q_s} + D_s$, we put the Lagrangian equations in the form (80.12). The coördinate q_s is said to be cyclic if the Lagrangian function L does not involve the coördinate q_s itself although it involves its velocity $\dot{q}_s$. When the dissipative force D_s is zero, we again have the momentum integral $p_s =$ constant, where $p_s = \frac{\partial L}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s}$.

Following the procedure introduced by Routh, we shall now emphasize the distinction between cyclic and noncyclic coördinates by using, as coördinates to describe the phase of the system, the position coördinates and momenta of the cyclic coördinates, together with the noncyclic coördinates and their velocities. In other words, we shall try to write the differential equations for the cyclic coördinates in the canonical form, keeping the original Lagrangian form for the noncyclic coördinates. It will be convenient to distinguish the cyclic coördinates from the others by a superposed bar. We shall suppose that we have r cyclic coördinates $\bar{q}$, and $m - r$ noncyclic coördinates q. On referring back to the argument in § 80, it will be seen that if we introduce the function $\bar{K}$ defined by the equation

$$\bar{K} = \Sigma_s \bar{p}_s \dot{\bar{q}}_s - T, \tag{81.3}$$

where the summation runs over the cyclic coördinates $\bar{q}$ only, we have

$$d\bar{K} = \Sigma_s\left(\dot{\bar{q}}_s d\bar{p}_s - \frac{\partial T}{\partial \bar{q}_s} d\bar{q}_s\right) - \Sigma_s\left(\frac{\partial T}{\partial \dot{q}_s} d\dot{q}_s + \frac{\partial T}{\partial q_s} dq_s\right) - \frac{\partial T}{\partial t}, \tag{81.4}$$

where T is supposed to be expressed in terms of the variables $\bar{q}$, $\bar{p}$, q, $\dot{q}$, and t, before the partial differentiations are performed. We are here tacitly supposing that the r momenta $\bar{p}$ are independent linear functions of the cyclic velocities, so that we can solve for these cyclic velocities in terms of the cyclic momenta, the cyclic coördinates, and the noncyclic coördinates and coördinate velocities. The single identity (81.4) is equivalent to the $2m + 1$ equations

$$\begin{gathered}\frac{d\bar{q}_s}{dt} = \frac{\partial \bar{K}}{\partial \bar{p}_s}, \quad \frac{\partial \bar{K}}{\partial \bar{q}_s} = -\frac{\partial T}{\partial \bar{q}_s}, \quad \frac{\partial \bar{K}}{\partial \dot{q}_s} = -\frac{\partial T}{\partial \dot{q}_s}, \\ \frac{\partial \bar{K}}{\partial q_s} = -\frac{\partial T}{\partial q_s}, \quad \frac{\partial \bar{K}}{\partial t} = -\frac{\partial T}{\partial t},\end{gathered} \tag{81.5}$$

so that our equations (78.3) may be replaced by the two sets of equations

$$\frac{d\bar{q}_s}{dt} = \frac{\partial \bar{K}}{\partial \bar{p}_s}, \quad \frac{d\bar{p}_s}{dt} = -\frac{\partial \bar{K}}{\partial \bar{q}_s} + Q_s, \quad (s = 1, 2, \cdots, r) \tag{81.6}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{K}}{\partial \dot{q}_s}\right) - \frac{\partial \bar{K}}{\partial q_s} = -Q_s. \quad (s = 1, 2, \cdots, m - r) \tag{81.7}$$

In the particular case where Q_s has the form $Q_s = -\frac{\partial V}{\partial q_s} + D_s$ for all the coördinates, both cyclic and noncyclic, these equations may be rewritten, on introducing the *modified Lagrangian function*

$$\bar{L} = -(\bar{K} + V). \tag{81.8}$$

In terms of the original Lagrangian function $L = T - V$ of (80.11) the modified function is given by

$$\bar{L} = L - \Sigma \bar{p}_s \dot{\bar{q}}_s. \tag{81.9}$$

The new form of the equations (81.6) and (81.7) is

$$\frac{d\bar{q}_s}{dt} = -\frac{\partial \bar{L}}{\partial \bar{p}_s}, \quad \frac{d\bar{p}_s}{dt} = \frac{\partial \bar{L}}{\partial \bar{q}_s} + D_s, \quad (s = 1, 2, \cdots, r) \tag{81.6a}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}_s}\right) - \frac{\partial \bar{L}}{\partial q_s} = D_s. \quad (s = 1, 2, \cdots, m - r) \tag{81.7a}$$

It will be observed that the cyclic coördinates have the Hamiltonian form, with $-\bar{L}$ taking the place of H, while the equations for the noncyclic coördinates have the Lagrangian form, with $+\bar{L}$ taking the place of L.

We have not yet made any use of the supposition that the coördinates $\bar{q}$ are cyclic. Here we have r momentum integrals $\bar{p}_s =$ constant; and these may be solved so as to enable us to express the velocities of the cyclic coördinates in terms of the position coördinates, the constant cyclic momenta, and the velocities of the noncyclic coördinates. The problem really reduces to the determination of the velocities of these noncyclic coördinates, and this is done by an integration of equations (81.7 *a*). Assuming that the dissipation forces D_s are zero, $\bar{L}$ will not involve $\bar{q}$; and so the only variables in $\bar{L}$ will be the noncyclic coördinates and their velocities. In other words, we may ignore the cyclic coördinates as far as the determination of the coördinate velocities goes. This is the reason for the name "ignorable coördinates" which is often applied to the cyclic coördinates.

EXAMPLES

1. A particle of mass m slides on a smooth horizontal rod which is in rotation around a vertical axis. Using polar coördinates, we have $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2)$, and the angular coördinate is cyclic. If there is no cyclic force, we have the momentum integral $mr^2\dot{\phi} = p_\phi =$ constant. Eliminating $\dot{\phi}$, we find for $\bar{L}$ the expression $\frac{1}{2}\left(m\dot{r}^2 + \frac{c^2}{mr^2}\right) - V - \frac{c^2}{mr^2}$, where c is the constant momentum and we are supposing V a function of r alone. The single Lagrangian equation for r is $m\ddot{r} - \frac{c^2}{mr^3} = -\frac{\partial V}{\partial r}$; and we see that, as far as determining r is concerned, the cyclic coördinate ϕ may be ignored. There is acting on the particle, in addition to possible applied forces, an apparent (centrifugal) force outward along the radius, of amount $\frac{c^2}{mr^3}$. The reader is urged to solve this problem directly by the method of § 35 (see § 30). This simple example is instructive as showing how what is called the potential energy of a system may be due to "concealed motions." Thus, if we were unaware of the rotating rod in this problem, and if there were no external applied forces acting on the particle, our observation of the motion would prompt us to write the differential equation for r in the form

$$m\ddot{r} = \frac{c^2}{mr^3} = -\frac{\partial}{\partial r}\left(\frac{c^2}{2\,mr^2}\right);$$

and we should therefore attribute a potential energy of amount $\frac{c^2}{2\,mr^2}$ to the system. Some physicists have been led by the consideration of examples such as this to the hypothesis that the potential energy of any dynamic system is really due to the kinetic energy of cyclic (and ignored) coördinates.

2. Four uniform rods, smoothly jointed at their ends, form a parallelogram which can move smoothly on a horizontal surface, one of the angular points being fixed. The configuration is initially rectangular, and the framework is set in motion in such a manner that the angular velocity of one pair of opposite sides is ω, that of the other pair being zero. Show that when the angle between the rods meeting in the fixed point is a minimum, the angular velocity of the system is ω.

Solution. This is a case of motion under no forces, so that the energy integral is $T =$ constant, the potential energy V being zero. In finding T we use the result that for a rigid body (see (55.4)), $T = \frac{1}{2} MV^2 + \frac{1}{2}(Ap^2 + Bq^2 + Cr^2)$. Here (p, q, r) are the components of the angular velocity along the principal axes of inertia of the rigid body at its center of mass or its fixed point (if it has a fixed point). V is either the velocity of the center of mass or else zero (if we are dealing with a rigid body having a fixed point). For a rod of length l turning about an end, the moment of inertia about any axis through this end and perpendicular to the rod is $\frac{Ml^2}{3}$, the moment about an axis passing through the rod being, of course, zero (see § 57).

In this example let the two rods meeting in the fixed point include an angle ϕ and have lengths $2\,a$ and $2\,b$, their masses being m_1 and m_4 respectively. Let the rod opposite the rod of mass m_1 have a mass m_3, the mass of the remaining rod being m_2, and let the rod of mass m_1 make an angle θ with a fixed direction in the line of motion. Then the kinetic energy of the first rod is $T_1 = \frac{2\,m_1a^2\dot{\theta}^2}{3}$, and that of the fourth is $T_4 = \frac{2\,m_4b^2(\dot{\theta}+\dot{\phi})^2}{3}$. The center of mass of the second rod has coördinates $[2\,a\cos\theta + b\cos(\theta+\phi),\ 2\,a\sin\theta + b\sin(\theta+\phi)]$, and its velocity components follow from these on time differentiation. The translational part of the kinetic energy of the second rod is thus found to be $\frac{m_2}{2}[4\,a^2\dot{\theta}^2 + b^2(\dot{\theta}+\dot{\phi})^2 + 4\,ab\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi]$. To this we have to add the rotational part $\frac{m_2b^2(\dot{\theta}+\dot{\phi})^2}{6}$ to get the complete kinetic energy T_2 of the second rod. Similarly, the kinetic energy T_3 of the third rod is found to be

$$T_3 = \frac{m_3}{2}[4\,b^2(\dot{\theta}+\dot{\phi})^2 + a^2\dot{\theta}^2 + 4\,ab\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi] + \frac{m_3a^2\dot{\theta}^2}{6}.$$

It is at once evident that the coördinate θ does not appear in the kinetic energy $T = T_1 + T_2 + T_3 + T_4$ of the system, so that θ is an ignorable coördinate, and we have the momentum integral p_θ a constant c. Since T is a homogeneous quadratic function of the coördinate velocities $\dot{\theta}$ and $\dot{\phi}$, the energy integral may be written in the form

$$\dot{\theta}\frac{\partial T}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial T}{\partial \dot{\phi}} = 2\,T = \text{constant}.$$

We are given the initial conditions $\dot{\theta} = \omega$, $\dot{\theta} + \dot{\phi} = 0$, $\phi = \frac{\pi}{2}$, so that initially $\frac{\partial T}{\partial \dot{\phi}} = 0$. Hence, on using the momentum integral in conjunction with the energy integral, we have $c\dot{\theta} + \dot{\phi}\frac{\partial T}{\partial \dot{\phi}} = c\omega$. When ϕ is a maximum or minimum, $\dot{\phi} = 0$ and $\dot{\theta} = \omega$.

3. Four equal uniform rods of length $2\,a$ are smoothly jointed so as to form a rhombus $ABCD$. The joint A is fixed, whereas C is free to move on a smooth vertical rod through A. Initially C coincides with A, and the whole system rotates around the vertical with angular velocity ω. Prove that if, in the subsequent motion, $2\,\alpha$ is the least angle between the upper rods, $a\omega^2\cos\alpha = 3\,g\sin^2\alpha$.

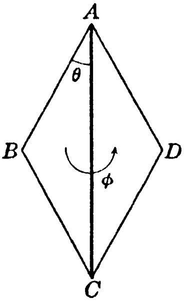

FIG. 32

Solution. The system requires for its description two angular coördinates θ and ϕ (see accompanying figure). The two upper rods have a fixed point A, and each has components of rotation $(\dot{\phi} \cos \theta, \dot{\phi} \sin \theta, \dot{\theta})$ along itself, and two mutually perpendicular directions at right angles to itself, respectively. The moment of inertia of either of the upper rods about an axis through A perpendicular to the rod is $\frac{4\,Ma^2}{3}$, so that the kinetic energy of the two upper rods is $\frac{4\,Ma^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)}{3}$. The Cartesian coördinates of the center of mass of either lower rod are $(a \sin \theta \cos \phi,\ a \sin \theta \sin \phi,\ 3\,a \cos \theta)$, from which the translational part of the kinetic energy of the lower rods follows at once. Adding their rotational energy, we get for their complete kinetic energy the result $\frac{Ma^2(\dot{\theta}^2 + \sin^2 \theta\, \dot{\phi}^2)}{3} + Ma^2[\sin^2 \theta\, \dot{\phi}^2 + \dot{\theta}^2(1 + 8 \sin^2 \theta)]$. The kinetic energy of the whole system is therefore

$$T = \frac{8\,Ma^2}{3}\,[\dot{\theta}^2(1 + 3 \sin^2 \theta) + \sin^2 \theta\, \dot{\phi}^2].$$

For the two upper rods we have the virtual work $2\,Mg\delta(a \cos \theta)$, and for the two lower $2\,Mg\delta(3\,a \cos \theta)$, so that the potential energy of the system is $V = -\,8\,Mag \sin \theta$. It follows at once that ϕ is a cyclic coördinate with the momentum integral $\sin^2 \theta\, \dot{\phi} = \omega$. The value of the constant momentum follows from the initial conditions $\theta = \frac{\pi}{2}$, $\dot{\phi} = \omega$. The energy integral, $T + V =$ constant, gives

$$a\dot{\theta}^2(1 + 3 \sin^2 \theta) + a \sin^2 \theta\, \dot{\phi}^2 = 3\,g \cos \theta + a\omega^2,$$

since initially $\dot{\theta} = 0$. When $\theta = \alpha$, we again have $\dot{\theta} = 0$; and we readily find $a\omega^2 \cos \alpha = 3\,g \sin^2 \alpha$.

4. Two uniform rods of length $2\,a$ are smoothly jointed to a fixed point. Upon them slides, by means of a smooth ring at each end, a third rod similar in all respects. Initially the three rods are in a horizontal line with the ends of the third rod at the middle points of the other two, and the rods are started rotating with angular velocity ω in a horizontal plane. Show that the third rod will slide off the other two unless $\omega^2 > \frac{2\,g}{\sqrt{3}\,a}$.

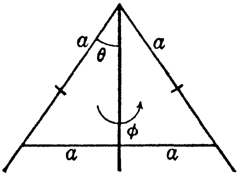

FIG. 33

Solution. The position of the system at any instant is completely specified by the two angles θ and ϕ of the figure. The center of mass of the third rod has a depth $a \cot \theta$, so that the translational part of its kinetic energy is $\frac{M}{2} a^2 \operatorname{cosec}^4 \theta\, \dot{\theta}^2$. The kinetic energy of the system is then readily found to be

$$T = \frac{4\, Ma^2}{3} (\dot{\theta}^2 + \sin^2 \theta\, \dot{\phi}^2) + \frac{Ma^2}{2} \operatorname{cosec}^4 \theta\, \dot{\phi}^2 + \frac{Ma^2 \dot{\phi}^2}{2}.$$

The potential energy of the system is $-2\, Mag \cos\theta - Mag \cot\theta$; and we see that ϕ is a cyclic or ignorable coördinate, the corresponding momentum integral being $(1 + 8 \sin^2\theta)\,\dot{\phi} = \text{constant} = 9\, \omega$, since initially $\theta = \frac{\pi}{2}$ and $\dot{\phi} = \omega$. The equation determining the minimum value of θ is found, as in Example 2, from the energy integral by substituting in it $\dot{\theta} = 0$ and using the momentum integral. We get

$$27\, a\omega^2 = (1 + 8 \sin^2\theta)[3\, a\omega^2 + 2\, g(\cot\theta + 2 \cos\theta)].$$

Upon simplifying this and dividing by $\cos\theta$, which factor corresponds to the value $\theta = \frac{\pi}{2}$, we find

$$f(\theta) = 12\, a\omega^2 \sin\theta \cos\theta - g(1 + 8 \sin^2\theta)(1 + 2 \sin\theta) = 0.$$

It is readily seen that $f\left(\frac{\pi}{2}\right) = -27\, g$, $f\left(\frac{\pi}{6}\right) = 6\left(a\omega^2 \frac{\sqrt{3}}{2} - g\right)$; so that if $a\omega^2 \frac{\sqrt{3}}{2} > g$, $f(\theta)$ has a zero between $\frac{\pi}{2}$ and $\frac{\pi}{6}$. Now by its very definition we know that $f(\theta)$ cannot change sign; for when multiplied by the positive number $\cos\theta$, it is a constant times $\dot{\theta}^2(1 + 3 \operatorname{cosec}^4\theta)$. Hence θ must alternate between $\frac{\pi}{2}$ and the nearest acute zero of $f(\theta)$. If this zero $< \frac{\pi}{6}$, the rod will not fall off; whereas if this zero $> \frac{\pi}{6}$, the rod will fall off the other two.

5. This is a problem on initial motions. A uniform rod of mass m_1 and length $2\, a$ is capable of rotating freely about its fixed upper extremity and is initially inclined at an angle $\frac{\pi}{6}$ to the vertical. A second rod of mass m_2 and of equal length $2\, a$ is smoothly attached to the lower end of the first, and rests initially at an angle $\frac{2\,\pi}{3}$ with it and in a horizontal position. Show that if the center of mass of the lower rod commences to move in a direction making an angle $\frac{\pi}{6}$ with the vertical, then $3\, m_1 = 14\, m_2$.

Solution. The coördinates θ and ϕ of the figure on the following page fix the position of the system at any instant. The center of mass of the lower rod has coördinates $(2\, a \cos\theta + a \cos\phi,\ 2\, a \sin\theta + a \sin\phi)$, so that the vertical and horizontal components of the velocity of this center of mass are

$$(-2\, a \sin\theta\, \dot{\theta} - a \sin\phi\, \dot{\phi}) \quad \text{and} \quad (2\, a \cos\theta\, \dot{\theta} + a \cos\phi\, \dot{\phi})$$

respectively. The translational part of the kinetic energy of the lower rod is therefore $\frac{m_2}{2}[4\,a^2\dot{\theta}^2 + 4\,a^2\dot{\theta}\,\dot{\phi}\cos(\theta+\phi) + a^2\dot{\phi}^2]$, and the kinetic energy of the system of two rods is

$$T = \frac{2\,m_1a^2\dot{\theta}^2}{3} + \frac{m_2a^2\dot{\phi}^2}{6} + \frac{m_2}{2}[4\,a^2\dot{\theta}^2 + 4\,a^2\dot{\theta}\,\dot{\phi}\cos(\theta+\phi) + a^2\dot{\phi}^2].$$

The potential energy V is $-\,m_1ag\cos\theta - m_2ag(2\cos\theta + \cos\phi)$. The direction of motion of the center of mass of the lower rod makes an angle with the downward vertical whose tangent is

$$-\frac{2\cos\theta\,\dot{\theta} + \cos\phi\,\dot{\phi}}{2\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi}}.$$

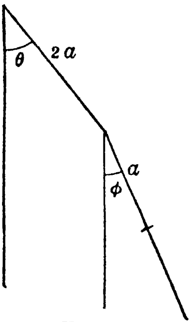

FIG. 34

This is to have initially the value $\frac{-1}{\sqrt{3}}$. Since the system starts from rest, $\dot{\theta}$ and $\dot{\phi}$ have initially the value zero, and this quotient becomes indeterminate. It is necessary to calculate by means of Lagrange's equations the initial values of the coördinate accelerations and to evaluate the indeterminate form by finding the ratio of the initial values of the time derivatives of the numerator and denominator. From the expression for T given above, and the fact that the initial values of $(\theta, \phi, \dot{\theta}, \dot{\phi})$ are $\left(\frac{\pi}{6}, \frac{\pi}{2}, 0, 0\right)$ respectively, it readily follows that the initial value of $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)$ is $\left(\frac{4\,m_1a^2}{3} + 4\,m_2a^2\right)\ddot{\theta}_0 + m_2a^2\ddot{\phi}_0$, and that the initial value of $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right)$ is $\frac{4\,m_2a^2\ddot{\phi}_0}{3} + m_2a^2\ddot{\theta}_0$, where $\ddot{\theta}_0$ and $\ddot{\phi}_0$ denote the initial values of the coördinate accelerations. Both $\frac{\partial T}{\partial\theta}$ and $\frac{\partial T}{\partial\phi}$ are initially zero, whereas the initial values of $\frac{\partial V}{\partial\theta}$ and $\frac{\partial V}{\theta\phi}$ are $ag\left(\frac{m_1}{2} + m_2\right)$ and agm_2 respectively. We find from the Lagrangian equations that $\frac{\ddot{\phi}_0}{\ddot{\theta}_0} = \frac{5\,m_1 + 18\,m_2}{4\,m_1 + 2\,m_2}$. As this should have the value 2, the required relation between m_1 and m_2 follows at once. That $\frac{\ddot{\phi}_0}{\ddot{\theta}_0} = 2$ follows from the fact that the indeterminate quotient above is $\frac{-\sqrt{3}\,\ddot{\theta}_0}{\ddot{\theta}_0 + \ddot{\phi}_0}$ and that this has the value $\frac{-1}{\sqrt{3}}$.

82. Application of Lagrange's equations to problems on impulsive forces. We have discussed impulsive forces in Chapter VIII and have seen that problems on impulsive forces are usually simpler than problems on finite or continuous forces; for we have only to integrate differential equations of the first order when we are dealing with impulsive forces, instead of differential equations of the second order as in problems on continuous forces. The fundamental theorem on impulsive forces acting on a dynamic system has been stated in § 59:

The change in the momentum localized-vector system is equivalent to the relative-impulse localized-vector system.

When we are dealing with a dynamic system subject to constraints, there will be, in general, impulsive reactions of these constraints when the system is subjected to the action of impulsive forces, and these reactions will have to be considered in the general case in calculating the impulse localized-vector system. However, if the system is a frictionless one, the virtual work of these impulsive reactions will be zero, and we may neglect them when investigating the motion of the system. If the position of the system is given by m generalized coördinates $(q_1, q_2, \cdots, q_m)$, we integrate the Lagrangian equations (78.3) with respect to the time between the limits t and $t + \Delta t$ and then let Δt become infinitesimally small. In this way we obtain the Lagrangian impulse equations

$$\Delta \frac{\partial T}{\partial \dot{q}_s} = \Gamma_s. \tag{82.1}$$

$\left(\text{Since } \frac{\partial T}{\partial q_s} \text{ is finite, the integral of this term vanishes when } \Delta t \rightarrow 0.\right)$ The symbol Δ before any quantity signifies the value of that quantity after the application of the impulsive forces less its value before the application of these forces. Γ_s is equal to $\int_t^{t+\Delta t} Q_s \, dt$ and is the generalized impulse corresponding to the coördinate q_s. In calculating Γ_s any forces which are not impulsive may be neglected, and in the time integration the position of the dynamic system may be regarded as fixed. The effect of the applied impulses is to change abruptly the coördinate velocities without any abrupt change in the coördinates themselves.

It may happen that the impulse communicated to the system is through a suddenly introduced constraint. Thus we may suppose that a certain number p of frictionless connections

$$f_r(q_1, q_2, \cdots, q_m) = 0 \quad (r = 1, 2, \cdots, p) \tag{82.2}$$

are introduced, so that our system has no longer m independent coördinates q after the application of the impulses. If, in addition to the introduction of the new constraints, other impulses are applied at the same time, our equations (82.1) must be replaced by the equations

$$\Delta \frac{\partial T}{\partial \dot{q}_s} = \Gamma_s + \lambda_1 \frac{\partial f_1}{\partial q_s} + \cdots + \lambda_p \frac{\partial f_p}{\partial q_s}, \quad (s = 1, \cdots, m) \quad (82.3)$$

where the λ's are undetermined multipliers. The argument necessary for the deduction of these equations is precisely the same as that given when deriving (78.6) and so is not repeated here. It may have been observed that the time is not indicated in the equations (82.2), whereas it is allowed for in equations (78.4). The reason for this is that the impulse equations serve merely to determine the change in the momenta of the system at the moment when the impulses are applied. The time variable is therefore a constant in these equations. The determination of the further motion of the system is an entirely separate problem. In other words, the application of impulsive forces suddenly changes the phase of the system, and the impulse equations serve to tell us what is the new phase. The ordinary Lagrangian equations (78.3), on the other hand, enable us to predict the whole phase history of the system, once the initial phase is given.

EXAMPLE

A circular disk impinges upon a fixed line in its plane and is then forced to roll upon it. Determine the velocity with which the disk rolls along the line (supposed to be smooth).

Solution. Taking the fixed line as one of a pair of rectangular Cartesian axes, with the origin of coördinates at the point where the circle strikes the line, the disk has three coördinates, namely, the coördinates (x, y) of its center and the turn θ of a line fixed in it with respect to the x-axis. The radius of the disk being a, we have for the kinetic energy T the expression $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{ma^2\dot{\theta}^2}{2}$. The sudden fixture is given by the two equations

$$y - a = 0, \quad x - a\theta = 0. \qquad (82.4)$$

Our impulse equations (82.3) are

$$\Delta m\dot{x} = \lambda_2, \quad \Delta m\dot{y} = \lambda_1, \quad \Delta \frac{ma^2\dot{\theta}}{2} = -a\lambda_2;$$

and the equations (82.4) of constraint give for the coördinate velocities after the fixture the relations $\dot{y} = 0$, $\dot{x} = a\dot{\theta}$. Denoting the initial velocities by the subscript zero, we have

$$\lambda_1 = -m\dot{y}_0, \quad \lambda_2 = m(\dot{x} - \dot{x}_0) = -\frac{m}{2}a(\dot{\theta} - \dot{\theta}_0).$$

On eliminating $\dot{\theta}$, we find

$$\dot{x} = \frac{2}{3}\left(\dot{x}_0 + \frac{a\dot{\theta}_0}{2}\right).$$

The disk will be suddenly stopped by its impact with the line if its motion before the impact is such that $\dot{x}_0 + \frac{a\dot{\theta}_0}{2} = 0$.

83. Carnot's theorem on impulsive forces. This is the analogue of the energy theorem for continuous forces. We shall consider a frictionless dynamic system, and shall further suppose that any suddenly introduced constraints, as well as those already existing, are *persistent,* that is, that the equations (82.2) defining the constraints do not involve the time variable explicitly. Upon integrating the equation (78.1) with respect to the time between the limits t and $t + \Delta t$, and then letting Δt become infinitesmally small, we get, as before, the equation

$$\left(\Delta\frac{\partial T}{\partial \dot{q}_1} - \Gamma_1\right)\delta q_1 + \cdots + \left(\Delta\frac{\partial T}{\partial \dot{q}_m} - \Gamma_m\right)\delta q_m = 0. \qquad (83.1)$$

Since the constraints are all supposed to be fixed, the actual kinematic-displacement differential is one of the infinity of virtual displacements; and so we may replace the δq_s of (83.1) by the corresponding $\dot{q}_s$. In this way we get

$$\left(\Delta\frac{\partial T}{\partial \dot{q}_1} - \Gamma_1\right)\dot{q}_1 + \cdots + \left(\Delta\frac{\partial T}{\partial \dot{q}_m} - \Gamma_m\right)\dot{q}_m = 0. \qquad (83.2)$$

Now the kinetic energy T is a quadratic function (homogeneous) of the coördinate velocities $\dot{q}$. Writing it out explicitly in the form

$$2\,T = \sum_{r,s} a_{rs}\dot{q}_r\dot{q}_s,$$

where each $a_{rs} = a_{sr}$, we have

$$\frac{\partial T}{\partial \dot{q}_r} = \sum_{s} a_{rs}\dot{q}_s,$$

so that (83.2) takes the form

$$\sum_{r,s} a_{rs}[\dot{q}_s - (\dot{q}_s)_0]\dot{q}_r = \sum_{r}\Gamma_r\dot{q}_r; \qquad (83.2\,a)$$

for the coefficients a_{rs}, being functions of the position coördinates q alone, are the same before and after the application of the impulses. Now the left-hand side of (83.2 a) can be put in the form

$$\frac{1}{2}\sum_{r,s}\{a_{rs}\dot q_r\dot q_s - a_{rs}(\dot q_r)_0(\dot q_s)_0 + a_{rs}[\dot q_r - (\dot q_r)_0][\dot q_s - (\dot q_s)_0]\}\ ; \qquad (83.3)$$

for since $a_{rs} = a_{sr}$, it is apparent that

$$\sum_{r,s} a_{rs}\dot q_r(\dot q_s)_0 = \sum_{r,s} a_{rs}(\dot q_r)_0\dot q_s.$$

We have, therefore, the equation

$$T - T_0 + T_{10} = \sum_r \Gamma_r\dot q_r, \qquad (83.4)$$

where T_{10} denotes what the kinetic energy would be if each coördinate velocity were replaced by the difference in value of this coördinate velocity before and after the application of the impulses. If the activity of the applied impulses is zero, — as, for example, when a smooth constraint is suddenly introduced, — the right-hand side of (83.4) is zero, and we have the special result

$$T_0 - T = T_{10}. \qquad (83.5)$$

This says that the introduction of one or more smooth fixed constraints causes a loss of kinetic energy of amount T_{10}. The more general Carnot theorem says that the activity of the applied impulses is equal to $T - T_0 + T_{10}$.

EXAMPLE

Six equal uniform rods form a regular hexagon loosely jointed at the angular points. A blow is given at right angles to one of them at its middle point. Show that the opposite rod begins to move with one tenth the velocity of the rod struck.

Solution. It is apparent from symmetry that the position of the system is completely fixed by a knowledge of the coördinates (x, θ) of the figure. In addition, it may be observed that in setting up the Lagrangian impulse equations we may put in the initial values of the position coördinates q before differentiating with respect to the $\dot q$, since the q's and $\dot q$'s are independent variables. Initially $\theta = \frac{\pi}{6}$; and we find $T = \frac{m}{2}\left(6\,\dot x^2 - 12\,a\dot x\dot\theta + 40\,a^2\frac{\dot\theta}{3}\right)$, m being the mass of each rod of length $2\,a$. The virtual work of the impulse I is $I\,\delta x$; and our equations (82.1)

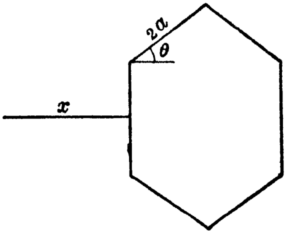

FIG. 35

become $\frac{\partial T}{\partial \dot{x}} = I$, $\frac{\partial T}{\partial \dot{\theta}} = 0$. The latter equation gives $\dot{x} = \frac{20\, a\dot{\theta}}{9}\cdot$ The x of the opposite rod is $x + 4\, a \cos\theta$, so that its initial velocity is $\dot{x} - 4\, a \sin\theta\, \dot{\theta}$, or $\frac{2\, a\dot{\theta}}{9}$, since initially $\theta = \frac{\pi}{6}\cdot$ This is the required result.

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EXERCISES

1. Two particles of masses m and m' are connected by an inextensible string passing through a smooth hole at the edge of a smooth horizontal table on which m rests. Determine the equations of motion of the particles and show that the tension of the strings is $T = \frac{mm'}{m + m'}\left(g + \frac{h^2}{m^2r^3}\right)$, where h is the constant value of $mr^2\dot{\theta}$.

2. Discuss the motion of two masses m_1 and m_2 connected by a light, inextensible string, and moving on smooth inclined planes whose angles with the horizontal are α and β respectively.

3. Discuss the motion of a double Atwood machine, where the mass m_2 is replaced by a smooth pulley of mass m_2 carrying masses m_3 and m_4.

4. A uniform rod is supported at its ends by two equal, inextensible, vertical cords suspended from fixed points. Find the small oscillations when the middle point moves vertically and the rod, remaining horizontal, turns around its middle point. (The angular motion is the same as that for small oscillations of a simple pendulum of length $\frac{l}{3}\cdot$)

5. A uniform rod of length $2a$ is supported symmetrically by two cords, each of length l, attached to its ends and to two fixed points at the same level distant $2(a + l\cos\alpha)$. Prove that the length of the simple equivalent pendulum, for small oscillations in the vertical plane through the cords, is $\frac{1}{3} al \sin\alpha \frac{1 + 3\sin^2\alpha}{a + l\cos^3\alpha}$.

6. A flywheel is connected by a crank and rod to a piston moving in a horizontal cylinder. When there is no steam in the engine, the flywheel rests in its position of equilibrium. Find the motion when displaced. Show that it is possible to balance the engine — that is, to arrange counterpoises, if necessary, on the wheel – so that the center of mass of the system remains at a constant height. When this is done, the kinetic energy of the system remains constant. Show that this does not imply a constant angular velocity of the wheel and crank.

7. A homogeneous sphere of radius a rolls down the outer surface of a fixed sphere of radius b without sliding. Find the motion.

8. A uniform rod, lying on a smooth horizontal table, is struck by a blow at one end. Determine the impulsive reaction at the other end required to keep it stationary.

9. Two equal uniform rods are hinged together and lie in a straight line on a smooth horizontal table. If the system is struck by a blow at one end, show that the initial angular velocity of the remote rod is one third that of the rod which is struck, and is oppositely directed.

10. A square is moving freely about a diagonal with angular velocity ω, when suddenly one of the angular points, not in that diagonal, becomes fixed. Determine the impulsive pressure on the fixed point, and show that the new angular velocity will be $\frac{\omega}{7}$.

CHAPTER XII

VIBRATIONS ABOUT A POSITION OF EQUILIBRIUM

84. The equilibrium of a dynamic system. A dynamic system is said to be in a state of equilibrium relative to a given reference frame if all its parts are and remain at rest relative to that frame. When the system is a *natural* one, — that is, when the equations of connection (77.2) do not involve the time variable explicitly, — it is easy to show that the necessary and sufficient condition for equilibrium is that the virtual work of the forces acting on the system should be zero. When the system is frictionless, so that the virtual work of the reactions of the (smooth) constraints is zero, these reactions may therefore be neglected in stating the criterion for equilibrium. That the vanishing of the virtual work is a necessary condition for equilibrium is rather obvious; for the total relative force on each particle of the system is, by definition, zero when the system is in equilibrium, and so the virtual work of the total relative-force system is zero. As the virtual work of the smooth constraints is zero, the virtual work of the applied forces must be zero also. The sufficiency of the condition follows from the fact that since the equations of connection do not involve the time explicitly, any hypothetical kinematic displacement is included in the infinity of virtual displacements. As each particle of the system starts from rest it must move (if it moves at all) in the direction of the total relative force acting on it; and so the virtual work, for this special virtual displacement, of the total relative-force system must be positive if the system moves at all. As the virtual work of the reactions is zero, this shows that for the particular virtual displacement furnished by the kinematic displacement differentials the virtual work of the applied-force system cannot be zero if the system moves from its position of rest. Hence

If the virtual work of the applied-force system is zero for all virtual displacements, the system is in a state of relative equilibrium.

When the applied-force system is derivable from a potential energy V, — that is, when we have the equations (80.5), where V is a function of $(q_1, q_2, \cdots, q_m)$ alone, — the vanishing of the virtual work for all virtual displacements implies the equations

$$\frac{\partial V}{\partial q_s} = 0, \quad (s = 1, 2, \cdots, m) \tag{84.1}$$

$(q_1, q_2, \cdots, q_m)$ being the independent coördinates of the system. Hence V has a stationary value at a position of equilibrium.

We now proceed to prove the stronger theorem, known as Dirichlet's theorem:

When V is actually a minimum, the position of equilibrium is a stable one.

By "stable" is meant that if the initial displacement and the initial coördinate velocities are made sufficiently small, — that is, if the change in phase from the equilibrium phase is made sufficiently small, — the phase of the system will remain *indefinitely* within an arbitrary but prescribed neighborhood of the equilibrium phase. The proof of Dirichlet's theorem depends on the existence of the energy integral

$$T + V = h. \tag{84.2}$$

To see how the fact that V is a minimum insures the stability of the equilibrium, let us so adjust the arbitrary additive constant, which is involved in V through its definition (80.5), that V has the value zero when the dynamic system is in its equilibrium position. Then, when the system is slightly displaced from this equilibrium position, V assumes, by hypothesis, a positive value. Let us denote the values of the coördinates q corresponding to the equilibrium position by q_0, and let us consider all positions of the system for which the numeric value of $q_s - (q_s)_0 \leqq \epsilon$ where $s = 1, 2, \cdots, m$ and ϵ is an arbitrarily assigned positive number. Then, when one of the coördinates (q_r, say) assumes its limiting value, so that the numeric value of $q_r - (q_r)_0 = \epsilon$, V will take values (positive) which will vary with the other coördinates q. Let the minimum value which V assumes, as these other coördinates vary within the limits assigned, be denoted by V_r. Then we know definitely that V_r is positive, provided ϵ has been taken small enough. Let $\bar{V}$ denote the least of the numbers $V_1, V_2, \cdots, V_m$ obtained in this way. From its definition, then, we know that if any one of the coördinates q differs from its equilibrium value by the amount ϵ, V will have a value greater than or equal to $\bar{V}$. Since T cannot be negative, from its definition, it follows from (84.2) that V is less than, or at most equal to, the energy constant h during any motion of the system. The equality sign holds only if the system comes to rest. The value of h in the equilibrium position being zero, we know that if we change the phase of the system by a sufficiently small amount, the motion starting with this phase will have its energy constant h as small as we wish. Let us make the initial phase with which the system is to start off so near the equilibrium phase that h is less than $\bar{V}$. Then it follows, from the definition of $\bar{V}$, that none of the coördinates q can, at any time during the subsequent motion, differ from its value in the position of equilibrium by an amount equal to ϵ. Since ϵ can be chosen arbitrarily, this means that the equilibrium is stable. During the entire motion near the equilibrium position, T takes values less than, or at most equal to, h; and this constant h may be made arbitrarily small by merely taking the initial phase of the motion sufficiently near the equilibrium phase.

85. Vibrations about a position of stable equilibrium. It will be convenient to change the position coördinates q, by the addition of constants if necessary, so that their values in the equilibrium position are each zero. During the entire motion which starts with a phase near the equilibrium phase, the coördinates q will then each take values less than an arbitrarily assigned number ϵ. (Of course, ϵ is chosen before the initial phase is determined.) We shall now determine the general characteristics of the motion of the system near the equilibrium position, under the distinct assumption that the number ϵ is so small that throughout the entire subsequent motion of the system the coördinates q and their velocities $\dot{q}$ are infinitesimals. By this we mean that we need keep only the lowest-order terms involving them in any expression. If the initial phase is not sufficiently near the equilibrium phase for this hypothesis to be satisfied, our results will not apply.

Since the potential energy V of the system is, by hypothesis, zero when the coördinates q have each the value zero, and since it has then a minimum value, the lowest-order terms in V must be quadratic terms; and, in accordance with what we have just said, it will not be necessary to consider terms in V of higher degree than the second. Then V is a quadratic function of the coördinates q; and since the position of equilibrium is assumed stable, this quadratic function must be definitely positive, that is, it cannot be negative, and it can have the value zero only when all the coördinates have the value zero. We shall find it unnecessary to use this property of the potential energy in the discussion; and so our remarks will apply to motions so far removed from the equilibrium phase that the potential energy may assume negative values, provided always that the q's and $\dot{q}$'s remain so small that they can still be regarded as infinitesimals. The kinetic energy T is a homogeneous quadratic function of the coördinate velocities $\dot{q}$, the coefficients being, in general, functions of the position coördinates q. In accordance with what has been said, it will be sufficient to take only the constant parts of these coefficients and to neglect the variable parts, as the latter are infinitesimal when compared with the constant parts. Since the coördinate velocities $\dot{q}$ are themselves infinitesimal, the part of T remaining when only the constant parts of the coefficients are retained is of the second order of infinitesimals, and the part of T neglected is of the third and higher orders of infinitesimals. The net result of these simplifications is that T and V are quadratic functions, with constant coefficients, of the coördinate velocities $\dot{q}$ and the coördinates q respectively. Writing them in the form

$$2\,T = \sum_{r,s} \alpha_{rs}\dot{q}_r\dot{q}_s, \quad (\alpha_{sr} = \alpha_{rs}) \tag{85.1}$$

$$2\,V = \sum_{r,s} \beta_{rs}q_rq_s, \quad (\beta_{sr} = \beta_{rs}) \tag{85.2}$$

we see that each of the partial derivatives $\dfrac{\partial T}{\partial q_s}$ vanishes, so that our equations (80.12) become

$$\sum_{r} (\alpha_{rs}\ddot{q}_r + \beta_{rs}q_r) = D_s. \tag{85.3}$$

We shall deal only with the cases (1) where all the dissipation forces $D_s = 0$, so that we have a conservative system, and (2) where each of the dissipation forces D_s is derivable from a dissipation function F, which is a quadratic function of the coördinate velocities $\dot{q}$. Exactly as with T, we may suppose the coefficients of this quadratic function to be constants; and, writing it in the form

$$2F = \sum_{r,s} \gamma_{rs}\dot{q}_r\dot{q}_s, \quad (\gamma_{sr} = \gamma_{rs}) \qquad (85.4)$$

our equations (85.3) take the form

$$\sum_r (\alpha_{rs}\ddot{q}_r + \gamma_{rs}\dot{q}_r + \beta_{rs}q_r) = 0, \quad (s = 1, 2, \cdots, m) \qquad (85.5)$$

since

$$D_s = -\frac{\partial F}{\partial \dot{q}_s}.$$

The reader will doubtless have observed that the discussion of the stability of the position of equilibrium was limited to the conservative case where all the dissipation forces are zero. The proof of stability hinged, then, on the existence of the energy integral $T + V = h$; but a rereading of the proof will show that an inequality $T + V < h$ would have done just as well for the purposes of the proof as the equality, h being the initial value of the energy. We saw, in (80.10), that $\frac{dH}{dt} = -2F$, where $H = T + V$ is the energy of the system. So it is necessary only to add the additional assumption that the dissipation function F is definitely positive in order to insure the stability of the equilibrium when the potential energy V is a minimum. Even when the position of equilibrium is not stable, our results will be valid so long as the assumption that the q's and the $\dot{q}$'s are infinitesimals is legitimate.

Equations (85.5) form a system of m linear differential equations of the second order, with constant coefficients, for the m independent coördinates q as functions of the time. In the conservative case the coefficients γ are all zero, and the coördinate velocities $\dot{q}$ do not enter these equations. We have here a problem which is a generalization of the problem of the single vibrating particle discussed in Chapter IV. We shall wish to take up problems where the forces acting on the system include, in addition to the conservative forces $-\frac{\partial V}{\partial q}$ and the dissipation forces $-\frac{\partial F}{\partial \dot{q}}$ already mentioned, certain external forces R_s whose values are, in general, quite independent of the position of the dynamic system and depend only on the time variable t. Our equations (85.5) must, then, be rewritten to read

$$\sum_r (\alpha_{rs}\ddot{q}_r + \gamma_{rs}\dot{q}_r + \beta_{rs}q_r) = R_s. \quad (s = 1, 2, \cdots, m) \qquad (85.6)$$

To distinguish the two cases we shall refer to the equations (85.5) as the equations governing the *free* vibrations of the system. If there are no dissipation forces, we shall refer to these vibrations as the free, undamped vibrations of the system. When there are dissipation forces, the vibrations, as will shortly appear, will be damped. Equations (85.6) will be referred to as the equations governing the *forced* vibrations of the system. If the forces R_s are constant, equations (85.6) may be satisfied by giving the q's constant values determined by the equations

$$\sum_r \beta_{rs} q_r = R_s. \quad (s = 1, 2, \cdots, n)$$

The effect of a constant force is to change the position of the equilibrium configuration.

It follows at once, from the linear character of equations (85.6), that if q is a solution for a certain set of external forces R, and if q' is a solution for another set of external forces R', then $q + q'$ will be a solution for the set of external forces $R + R'$. For this reason the mode of procedure is first to find the most general solution of equations (85.5) governing the free vibrations, and then to find a single particular solution of equations (85.6) governing the forced vibrations. The most general forced vibration will then be found by adding the general free vibration to the single particular forced vibration. In finding the particular forced vibration we avail ourselves of the linear character of equations (85.6) to analyze the external force R into two simpler parts R_1 and R_2. We are allowed to add the two solutions corresponding to the applied forces R_1 and R_2 separately. We shall now discuss in some detail the problem of the free vibrations.

86. The free vibrations of a dynamic system about a position of stable equilibrium. The differential equations (85.5) governing the problem may be regarded as a set of m linear algebraic equations for the m coördinate accelerations $\ddot{q}_s$ in terms of the m coördinate velocities $\dot{q}$ and the coördinates q themselves. It is possible to solve these equations, and thus express the coördinate accelerations explicitly in terms of the coördinates and their velocities, provided the determinant of the coefficients α_{rs} of the accelerations does not vanish. We shall suppose that this condition is fulfilled, and shall merely say that if the determinant of the coefficients α_{rs} does happen to be zero, the coördinates q adopted are not suitable for the discussion of the small motions about the position of stable equilibrium. The way out of the difficulty is to introduce another set of coördinates.

To show that the difficulty referred to may arise even in the discussion of quite simple problems, let us glance at the problem of the small oscillations of a particle, under the influence of gravity, on a sphere about the position of stable equilibrium of the particle — the lowest point of the sphere. The radius of the sphere being a, it is natural to choose, as our coördinates q_1 and q_2, the two angles θ and ϕ of a system of polar coördinates whose origin is at the center of the sphere and whose polar axis is vertically downward. We have, then, $T = \frac{ma^2}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$; so that $\alpha_{11} = ma^2$, $\alpha_{12} = \alpha_{21} = 0$, $\alpha_{22} = 0$ (since in the position of equilibrium $\theta = 0$). Hence the coördinates adopted are not suitable, and the reason in this particular case is quite obvious. The position of stable equilibrium is given by $\theta = 0$, without reference to the value of the second coördinate ϕ, which is, in fact, undefined at the position of equilibrium. The way out of the difficulty is to use as our coördinates (q_1, q_2) the Cartesian coördinates (x, y) of the projection, on the tangent plane at the lowest point of the sphere, of the position of the moving particle. We now have $2\,T = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, with $z^2 = a^2 - x^2 - y^2$. Since $z\dot{z} = -(x\dot{x} + y\dot{y})$, and since z is approximately equal to a, we see that $\dot{z}$ is of a higher order of infinitesimals than $\dot{x}$ or $\dot{y}$. We have, then, in accordance with the general discussion of § 85, $2\,T = m(\dot{x}^2 + \dot{y}^2)$, $V = mg(a - z)$. It is necessary to expand V in terms of (x, y) near the point $(0, 0)$, and to do this we observe that at $(0, 0)$ the value of z is a. Its first derivatives have at this point the values $(0, 0)$, and its second derivatives have the values $\left(-\frac{1}{a}, 0, -\frac{1}{a}\right)$. Hence $V = \frac{mg(x^2 + y^2)}{2\,a}$ to the order of approximation adopted; and our equations (85.5) become

$$\ddot{x} = \frac{-gx}{a}, \quad \ddot{y} = \frac{-gy}{a},$$

showing that the projection of the particle on a horizontal plane is in general an ellipse (see § 39).

Returning now to a consideration of equations (85.5), we make (for reasons similar to those given in § 38, where we were discussing the analogous problem for a single particle) the trial solution $q_s = A_s e^{\lambda t}$ $(s = 1, 2, \cdots, m)$. Here the $m + 1$ quantities A_s and λ are constants, possibly complex, whose values we shall now proceed to investigate. Our differential equations (85.5) are replaced by the m algebraic equations

$$\sum_r (\alpha_{rs}\lambda^2 + \gamma_{rs}\lambda + \beta_{rs})A_r = 0. \quad (s = 1, 2, \cdots, m) \quad (86.1)$$

These are linear, homogeneous equations for the undetermined constants A_s, and we know from the elementary theory of such equations that they cannot have any solution other than the trivial and obvious one, $A_s = 0$ for every s, unless the determinant of the coefficients happens to be zero. We avail ourselves of the fact that the remaining undetermined constant λ appears in this determinant of the coefficients, to choose its value so that this determinant will be zero. Denoting the determinant by $D(\lambda)$, we have first to solve the algebraic equation $D(\lambda) = 0$ of degree $2\,m$ in λ. (That this equation is in all cases actually of degree $2\,m$, and never of lower degree, follows from the fact that the coefficient of $\lambda^{2\,m}$ is the determinant of the coefficients α_{rs} of the quadratic expression in the $\dot{q}$'s giving the kinetic energy; and we have made the definite hypothesis that this determinant is not zero.) After substituting any one of the values of λ, which make $D(\lambda) = 0$, in equations (86.1), these equations are no longer independent, and it is possible to express at least one of them as a linear combination of the others. This tells us that we cannot determine the actual values of the A's since we are not in possession of a sufficient number of independent equations. At least one of the quantities A_s may be chosen arbitrarily, and then the remaining A's may be expressed linearly in terms of these arbitrary ones. Thus, if only one of the A's is arbitrary, we may say that the A's are not determined but that their ratios are. We should then have, associated with each zero of the determinant $D(\lambda)$, a solution of our differential equations (85.5) involving one arbitrary constant. If $D(\lambda)$ has $2\,m$ distinct zeros, we should in this way — on combining the solutions obtained, one for each value of λ — secure a solution of equations (85.5) involving $2\,m$ arbitrary constants; and this must be the general solution, since it has the correct number of arbitrary constants. But it is quite evident that matters might not turn out so simply as all this. For example, the equation $D(\lambda) = 0$ may well have repeated roots; and, corresponding to any root, it is conceivable that more than one of the unknown constants A_s may be chosen arbitrarily.

Certain general theorems as to the nature of the roots of the equation $D(\lambda) = 0$, which is known as the *characteristic* equation, are readily derived when there are no dissipation forces. Our equations (86.1) may then be written in the form

$$\lambda^2 \sum_r \alpha_{rs} A_r + \sum_r \beta_{rs} A_r = 0. \quad (s = 1, 2, \cdots, m) \tag{86.2}$$

Let us multiply the first of these equations by $\bar{A}_1$, the conjugate complex quantity to A_1, and so on for each of the m equations and let us add then all the results so obtained. We find

$$\lambda^2 \sum_{r,\,s} \alpha_{rs} A_r \bar{A}_s + \sum_{r,\,s} \beta_{rs} A_r \bar{A}_s = 0. \tag{86.3}$$

Each of the double summations is equal to its conjugate (on account of the symmetric character of the coefficients α_{rs} or β_{rs}) and is therefore necessarily real. Hence λ^2 is either real or indeterminate (the latter possibility occurring when each of the double summations in (86.3) is zero). Writing, for the moment, A_r explicitly in the form $a_r + ib_r$, to show its real and imaginary parts, we have $\sum_{r,\,s} \alpha_{rs} A_r \bar{A}_s = \sum_{r,\,s} \alpha_{rs}(a_r a_s + b_r b_s)$, since it is real. Neither of the double sums $\sum_{r,\,s} \alpha_{rs} a_r a_s$ and $\sum_{r,\,s} \alpha_{rs} b_r b_s$ can be zero unless all the a's or b's, respectively, are zero. For the first double sum, for example, is the double kinetic energy of the system corresponding to the coördinate velocities $\dot{q}_s = a_s$ $(s = 1, 2, \cdots, m)$; and the kinetic energy is definitely positive unless the system is at rest. Hence the double sum $\sum_{r,\,s} \alpha_{rs} A_r \bar{A}_s$ of (86.3) is positive, except where all the A's $= 0$, — a case which does not arise, since we have definitely chosen λ so that the equations (86.2) possess a nontrivial solution. We are therefore assured that λ^2 is real, and hence the ratios of the A's determined by the equations (86.2) are real. For these ratios are given by the ratios of determinants formed from the coefficients in equations (86.2), and these coefficients are now seen to be real. Furthermore, a similar argument shows that the second double summation, $\sum_{r,\,s} \beta_{rs} A_r \bar{A}_s$, is positive. The argument hinges on the fact that V has the value zero when the system is in the equilibrium position, and that it has a minimum there, the equilibrium position being assumed stable. Hence, for any position other than the equilibrium position but in the neighborhood of that position, V has a definitely positive value. Our equation (86.3) tells us that λ^2 is not only real but negative. In the case, then, where there are no dissipation forces, so that an energy integral exists, the characteristic equation of degree 2 m in λ may be regarded as an equation of degree m in λ^2, and we know that all its roots are real and negative. All the values of λ are pure imaginaries, and associated with each one is its conjugate (negative). With each set of A's corresponding to a certain value of λ there may be associated the *same* set of A's corresponding to the negative value of λ (since λ enters equations (86.2) only through its square). As, however, the A's are not uniquely determined by the value of λ, we must mean by the *same* set any set differing from the original one — if at all — only in respect to this indeterminateness. Our general solution of the differential equations governing the free, undamped vibrations will therefore be of the form

$$q_s = \sum_r C_{sr} \cos(\mu_r t - \delta_r), \quad (s = 1, 2, \cdots, m) \tag{86.4}$$

where μ_r is the positive square root of $-\lambda_r^2$. The constants δ_r depend only on the subscript r, and not on s. For, corresponding to the two roots $\lambda_r = i\mu_r$ and $\bar{\lambda}_r = -i\mu_r$ of the characteristic equation, we have the two sets of solutions $q_s = A_s e^{i\mu_r t}$ and $q'_s = A'_s e^{-i\mu_r t}$ of our differential equations. We have just pointed out that $A'_s = m_r A_s$ for every s; the multiplier m_r not varying with s, since the ratios of the $A''s$ are the same as those of the A's. Combining the two sets of solutions, q_s and q'_s, we obtain the solution

$$q''_s = A_s(e^{i\mu_r t} + m_r e^{-i\mu_r t}).$$

The ratios of the A's being real, we may write $A_s = C_s P_r$, where C_s is real and P_r does not vary with s; then

$$q''_s = C_s(P_r e^{i\mu_r t} + m_r P_r e^{-i\mu_r t}).$$

The quantity δ_r is determined completely by the expression in parentheses, and this does not vary with s. The n^2 amplitudes C_{sr} are not independent. In the general case their ratios, for a given value of r, are determined by the corresponding value of λ_r^2 or, what is the same thing, of μ_r. However, this general and rather casual discussion is not sufficient to take care of exceptional circumstances, such as the possibility of the equality of two or more zeros λ of the characteristic equation, and it is often the special or exceptional cases which are of most interest. A more detailed investigation will be considerably simplified by the introduction of what are known as "normal" coördinates, whose characteristic feature is that the variables are separated in the quadratic expressions for the kinetic and potential energies. In the next paragraph we shall give an elementary mathematical and constructive proof of the possibility of finding such coördinates in the general case.

87. The simultaneous reduction of the kinetic and potential energies to sums of square terms. The kinetic and potential energies do not involve the same variables, the velocities $\dot{q}$ being the variables in the kinetic energy, and the coördinates q themselves being the variables in the potential energy. However, the only change we shall find it necessary to make in the coördinates, for our purpose, will be a linear transformation of the type

$$q'_s = a_{s1}q_1 + a_{s2}q_2 + \cdots + a_{sm}q_m, \tag{87.1}$$

with constant coefficients a_{sr}. The coördinate velocities $\dot{q}$ will accordingly change in the same way:

$$\dot{q}'_s = \sum_r a_{sr}\dot{q}_r. \quad (s = 1, 2, \cdots, m)$$

The fact that the variables in the two energies are different will not, therefore, be of any importance, and we shall find it convenient, in expounding the general theory, to consider two quadratic forms

$$F_1 = \sum_{r,s} \alpha_{rs}x_r x_s, \quad F_2 = \sum_{r,s} \beta_{rs}x_r x_s$$

in the same variables x_r $(r = 1, 2, \cdots, m)$. There is no lack of generality in supposing the forms so written that $\alpha_{rs} = \alpha_{sr}$, $\beta_{rs} = \beta_{sr}$; and we shall imagine this done. The form F_1 will correspond to the kinetic energy, and the form F_2 to the potential energy. Now, if we have a single quadratic form $F = \sum_{r,s} \epsilon_{rs}x_r x_s$, the determinant of the coefficients ϵ_{rs} is called the discriminant of the form; and we shall denote it by ϵ. When the discriminant of the form vanishes, the form is said to be singular, and the physical meaning of the singularity of a quadratic form is that it may be expressed in terms of a lesser number of variables than those actually in use (the new variables being supposed to be connected with the old by a linear transformation of the type (87.1)). To see this, let us suppose that we introduce new variables y by means of the equations

$$x_s = \sum_r a_{sr} y_r. \quad (s = 1, 2, \cdots, m) \tag{87.2}$$

Then, by direct substitution, F takes the new form

$$F = \sum_{p,q} \sigma_{pq} y_p y_q,$$

where

$$\sigma_{pq} = \sum_{r,s} \epsilon_{rs} a_{rp} a_{sq}. \tag{87.3}$$

The new discriminant being a determinant of which σ_{pq} is the element in the pth row and the qth column, it follows that it is a product of three determinants. In fact, when we multiply two determinants, we get a new determinant of which the element in the rth row and sth column is the scalar product of the elements of the rth row of the first and the elements of the sth column of the second. If we denote the determinant of the transformation (87.2) by a, we may denote the same determinant, with its rows and columns interchanged, by a'; so that a' is equal to a in value, but is different in appearance. Then the summation with respect to r in (87.3), that is, $\sum_r a_{rp} \epsilon_{rs}$, may be written as $\sum_r a'_{pr} \epsilon_{rs}$, so that it is the element in the pth row and sth column of the determinant product $a'\epsilon$. Hence the double summation of (87.3) is the element in the pth row and qth column of the determinant product of $a'\epsilon$ and a. Since a' is equal in value to a, this shows that the determinant σ of the new form of F is equal to ϵ multiplied by a^2. We shall limit ourselves to transformations (87.2) that can be solved for the y's in terms of the x's, which is equivalent to saying that we shall suppose the new coördinates y, introduced by (87.2), to be linearly independent of each other. Mathematically this means that the determinant a of the transformation is not zero, and such transformations of coördinates are called reversible. We see, then, that if the discriminant of a quadratic form is zero in one set of coördinates, it is necessarily zero in any other set derived from the original set by a reversible transformation. Similarly, if it is not zero in the original set, it cannot be zero in the new set. In a word, the singularity or nonsingularity of a quadratic form is an absolute property of the form; that is, it is independent of the particular coördinates adopted to describe the form. It may be observed that if we follow a reversible transformation (87.2) by a second reversible transformation

$$y_s = \sum_r b_{sr} z_r, \quad (s = 1, 2, \cdots, m)$$

the single transformation from x to z is

$$x_s = \sum_r c_{sr} z_r, \quad (s = 1, 2, \cdots, m)$$

where

$$c_{rs} = \sum_p a_{rp} b_{ps}.$$

The determinant of this transformation is one whose element in the rth row and sth column is the scalar product of the rth row of a and the sth column of b, and so the value of this determinant is ab. This tells us that a succession of two (and hence of any number of) reversible transformations may be represented as a single reversible transformation. It is now easy to see how to build up a reversible transformation which will separate the variables in any quadratic form $F = \sum_{r,s} \epsilon_{rs} x_r x_s$. If there is one square term in F whose coefficient is not zero (usually there will be many such), we may denote this nonzero coefficient by ϵ_{pp}. We then make the transformation

$$y_p = \sum_k \epsilon_{pk} x_k, \quad y_s = x_s, \quad (s \neq p) \tag{87.4}$$

which is reversible $\left(\text{its determinant being } \dfrac{1}{\epsilon_{pp}}\right)$; and we see that

$$F = \frac{y_p{}^2}{\epsilon_{pp}} + G,$$

where G is a quadratic form *which does not involve the variable* y_p. Continuing this method, F can be reduced to a sum of square terms by a succession of reversible transformations (and hence by a single reversible transformation), *unless* it happens that at some stage the quadratic form G does not possess any square terms. Let us suppose, then, that G is of this form, and let δ_{ij} be a nonzero coefficient of it. If we make the reversible transformation

$$z_i = \sum_r \delta_{ir} y_r, \quad z_j = \sum_p \delta_{jp} y_p, \quad z_t = y_t, \quad (t \neq i \text{ or } j)$$

the product $2\, z_i z_j$ contains, as terms involving y_i or y_j, the corresponding terms of G multiplied by δ_{ij}, so that

$$G = \frac{2\, z_i z_j}{\delta_{ij}} + H,$$

where H does not involve either of the variables z_i or z_j. The additional reversible transformation

$$z_i = u_i + u_j, \quad z_j = u_i - u_j, \quad z_t = u_t, \quad (t \neq i \text{ or } j)$$

reduces G to the form

$$G = \frac{2(u_i^2 - u_j^2)}{\delta_{ij}} + H,$$

where H does not involve the variables u_i or u_j. In this way, then, a quadratic form can be reduced by a reversible transformation of coördinates to a sum of square terms only, and the transformation is by no means unique. The discriminant of the form, when expressed in terms of the new coördinates, is merely the product of the coefficients of the square terms, as all but the diagonal terms of the determinant of the coefficients are zero. Hence, if a form is singular, one at least of the coefficients of the square terms must be zero, and this variable will therefore not appear in the form at all. This justifies the earlier remark as to the physical meaning of a form's being "singular." A quadratic form is said to be definite if it always has the same sign, and if it has the value zero only when all the variables in it are zero. It follows that when the form is reduced to a sum of squares, the coefficients must all be of the same sign if the form is to be definite. If the form is positively definite, all the coefficients of the squared terms must be positive; and if negatively definite, all these coefficients must be negative. It is, in fact, easy to see that if a form is positively definite, the coefficients of the square terms must all be positive, even when the product terms have not been removed. If, for example, the coefficient of the x_p^2 term is zero or negative, the system of values $x_p = 1$, $x_s = 0$, where $s \neq p$, would make the form zero or negative, as the case may be, and this would be contrary to our hypothesis as to the positively definite character of the form.

In order to discuss effectively the question of the reduction of *two* quadratic forms

$$F_1 = \sum_{r,s} \alpha_{rs} x_r x_s, \quad F_2 = \sum_{r,s} \beta_{rs} x_r x_s \tag{87.5}$$

simultaneously to a sum of square terms, it is convenient to consider the "pencil" of quadratic forms

$$F = F_2 - \lambda F_1, \tag{87.6}$$

where λ is an arbitrary parameter (independent in value of the coördinates x). The discriminant of the quadratic form F, set equal to zero, furnishes what is known as the characteristic equation (or λ-equation) of the pair of forms F_1 and F_2. It is of degree m in λ if F_1 is nonsingular, as the coefficient of λ^m in the discriminant of F is plus or minus the discriminant of F_1. In the applications we shall make of this theory, F_1 is the kinetic energy. This, being positively definite, is necessarily nonsingular. The argument given in § 86 assures us that when F_1 is a definite form, the roots of the characteristic equation are all real. Denoting one of these roots by λ_1, we rewrite the pencil of quadratic forms (87.6) as follows:

$$F_2 - \lambda_1 F_1 + (\lambda_1 - \lambda)F_1, \tag{87.6a}$$

and observe that the quadratic form $F_2 - \lambda_1 F_1$ is singular, since, by the definition of λ_1, its discriminant is zero. We can therefore find a reversible transformation of the type (87.2) in such a way as to express $F_2 - \lambda_1 F_1$ in terms of a less number of variables than the original number m. Calling the new coördinates thus introduced $(y_1, y_2, \cdots, y_m)$, our pencil (87.6 a) takes the form

$$(\lambda_1 - \lambda)\bar{F}_1(y_1, y_2, \cdots, y_m) + \phi(y_2, y_3, \cdots, y_m), \tag{87.7}$$

where at least one of the new variables (y_1, say) does not appear in the quadratic form ϕ which is the transformation of the singular form $F_2 - \lambda_1 F_1$. The bar is placed over F_1 to indicate that although $\bar{F}_1$ is equal to F_1, its appearance is usually quite different. Since λ is an arbitrary parameter, and since (87.7) is merely another way of writing (87.6), we must have the terms in (87.6) which do not involve λ equal to the terms in (87.7) which are independent of λ, and similarly for the terms involving λ. As $\bar{F}_1$ is a positively definite form, the coefficient of y_1 in it is positive, and we can find a reversible transformation from y to z such that $\bar{F}_1$ takes the form $z_1^2 + \xi(z_2, z_3, \cdots, z_m)$ (see (87.4)). This same transformation will send $\phi(y_2, y_3, \cdots, y_m)$ into a function $\bar{\phi}(z_2, z_3, \cdots, z_m)$. So (87.7) takes the new form

$$(\lambda_1 - \lambda)[z_1^2 + \xi(z_2, z_3, \cdots, z_m)] + \bar{\phi}(z_2, z_3, \cdots, z_m). \tag{87.7a}$$

Comparing this with (87.6), we obtain the *two* equations

$$\begin{aligned} F_2 &= \lambda_1 z_1^2 + \eta(z_2, z_3, \cdots, z_m), \\ F_1 &= z_1^2 + \xi(z_2, z_3, \cdots, z_m), \end{aligned} \tag{87.8}$$

where we have grouped the two terms $\lambda_1\xi + \bar{\phi}$ into the single term η. It is at once apparent that the quadratic form ξ in the $m-1$ variables $(z_2, z_3, \cdots, z_m)$ must be positively definite; for if there existed a set of nonzero values of these coördinates making ξ either negative or zero, we should have, by merely adding to this set the value $z_1 = 0$, a set of nonzero values of the m variables $(z_1, z_2, \cdots, z_m)$ making F_1 either negative or zero, and this is impossible, since F_1 is positively definite. We may now form the pencil $\eta - \lambda\xi$ of quadratic forms in the $m-1$ variables $(z_2, z_3, \cdots, z_m)$ and proceed in exactly the same manner as we have done with the pencil (87.6). Continuing in this way, we finally obtain the desired reduction of the two forms F_1 and F_2 to a sum of square terms

$$\begin{aligned} F_1 &= z_1^2 + z_2^2 + \cdots + z_m^2, \\ F_2 &= \lambda_1 z_1^2 + \lambda_2 z_2^2 + \cdots + \lambda_m z_m^2. \end{aligned} \tag{87.9}$$

The characteristic equation, or λ-equation, of the two forms, when these new variables are used, is at once seen to be

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_m - \lambda) = 0, \tag{87.10}$$

showing that the roots of the characteristic equation are the coefficients of F_2 when both forms are reduced to sums of squares, the coefficients of F_1 being all unity. (These roots cannot depend on the particular coördinates in use, since the vanishing of the discriminant of a quadratic form is an absolute property, that is, is not affected by a change of coördinates.)

88. Properties of the characteristic numbers λ_s. The coefficients λ_s of F_2 are, as has just been remarked, the roots of the characteristic equation of the two forms, and they may be called the characteristic numbers of the two quadratic forms. If we try to find values of the variables z which make one of the forms (87.9) stationary, — that is, a minimum, maximum, or minimax, — subject to the condition that the other of the two forms remains constant, we set the differentials of each of the two forms equal to zero. We find m possible solutions, in each of which all the variables but one are zero, the corresponding stationary value of F_2, say, being $\lambda_s C$, where $s = 1, 2, \cdots, m$ and where C is the constant value of F_1. Geometrically we may, if we like, use the language of hyperspace and regard F_1 as the square of the distance of the point z from the origin, and we see that the question here is that of determining the axes of a second-degree surface (see § 9). Let us now fix $C = 1$; and let us seek the maximum value taken by F_2 when, in addition to the condition $F_1 = 1$ on the variables $(z_1, z_2, \cdots, z_m)$, we have h arbitrary linear homogeneous relations among these variables, where $h = 1, 2, \cdots, m$. We shall, for convenience, assume $\lambda_1, \lambda_2, \cdots, \lambda_m$ to be arranged in descending order of magnitude. If we set all the z's from z_{h+1} on to z_m equal to zero, the remaining variables will be determined by the h homogeneous linear relations and the single nonhomogeneous relation $F_1 = 1$, the homogeneous relations merely determining the ratios of the h variables z_1 to z_h. For this set of values of the variables z, F_2 will take the value $\lambda_1 z_1^2 + \cdots + \lambda_h z_h^2$. Since the λ's are arranged in descending order of magnitude, this expression is greater than or equal to $\lambda_h(z_1^2 + \cdots + z_h^2)$, or λ_h, since the expression in parenthesis, being equal to F_1, for the values of the variables chosen, has the value unity. Hence, no matter what the linear, homogeneous relations of the variables are, the maximum value of F_2 is certainly as large as, and usually larger than, λ_h. In the special case where the linear relations are $z_1 = 0, z_2 = 0, \cdots, z_{h-1} = 0$, z_h^2 is equal to unity, and F_2 has exactly the value λ_h. We have, therefore, the following rather useful characterization of the coefficients λ_h of F_2:

When, in addition to the relation $F_1 = 1$, *we add* $h - 1$ *linear, homogeneous restrictions on the variables,* F_2 *will have an absolute maximum subject to these restrictions, and this maximum will be a function of these restrictions or constraints. The least value that this maximum can take, as we vary the constraints without changing their number, is exactly* λ_h, *where* $h = 1, 2, \cdots, m$.

If, now, we use constraints (k in number) to reduce the number of variables, we again have characteristic numbers of the two forms in the lesser number of variables; and we may, for the moment, denote these characteristic numbers by $\bar{\lambda}_1, \cdots, \bar{\lambda}_p$, where $p = m - k$. It follows at once, from the foregoing characterization of the coefficients λ and $\bar{\lambda}$, that

$$\lambda_q \geqq \bar{\lambda}_q \geqq \lambda_{q+k}, \quad (q = 1, 2, \cdots, p). \tag{88.1}$$

For when the constraints are imposed, the choice of variables in finding the maxima of F_2 *subject* to q further relations is not so great as before the number of independent variables was lessened, so that the maxima of F_2 are certainly not greater than the original maxima before the constraints were imposed. The absolute minimum of these maxima, as the q arbitrary linear, homogeneous relations of the coördinates are altered, is therefore less than or equal to the previous absolute minimum of the maxima. In other words, $\bar{\lambda}_q \leqq \lambda_q$. However, the restrictions on the choice of the variables, when $k + q - 1$ *arbitrary* linear, homogeneous relations are imposed on the variables, include as a special case the restriction imposed by k fixed and $q - 1$ arbitrary constraints, and so the maxima of F_2 in the latter case must be less than or equal to the corresponding maxima in the former. This leads to the result that the minimum of the absolute maxima as the $q - 1$ constraints are varied must be less than or equal to the minimum of the maxima as all the $k + q - 1$ constraints are varied. In other words, $\bar{\lambda}_q \leqq \lambda_{q+k}$.

The content of this result may be better understood if we consider the case of three variables, $m = 3$, and use the geometric interpretation of the algebra. We are, then, originally finding the maximum, minimum, and minimax values of the squared distance of a point on a second-degree surface from its center. For definiteness we shall suppose the second-degree surface to be an ellipsoid. (In this interpretation we are really making F_1 stationary, subject to the relation $F_2 = 1$; but the results are the same as if we were making F_2 stationary, subject to the relation $F_1 = 1$. In each case we have to equate to zero the differential of each of the quadratic forms F_1 and F_2.) Putting a single linear, homogeneous constraint on the coördinates is interpreted by saying that the point we are interested in must lie not only on the ellipsoid but also on a plane through the center of the ellipsoid; that is, the point is confined to an ellipse, the plane section of the ellipsoid by the given central plane. Our result indicates that the major axis of this ellipse must be less than or equal to the greatest axis of the ellipsoid and greater than or equal to the mean axis of the ellipsoid, and that the minor axis of the ellipse must be less than or equal to the mean axis of the ellipsoid and greater than or equal to the least axis of the ellipsoid. However, the real interest in the theorem is that it gives at once a proof of a theorem, due to Rayleigh, on the change in the periods of vibration of a dynamic system performing small oscillations when additional constraints are imposed on the system.

89. The free normal, undamped vibrations of a dynamic system about a position of stable equilibrium. We have seen that near a position of stable equilibrium the potential energy of the system, as well as the kinetic energy, is a positive definite form, the former in the coördinates and the latter in the coördinate velocities. Hence the characteristic numbers of the two forms are all positive; and we shall emphasize this by writing $\lambda_r = \mu_r^2$, where $r = 1, 2, \cdots, m$. In terms of the normal coördinates q we have

$$2\,T = \dot{q}_1^2 + \dot{q}_2^2 + \cdots + \dot{q}_m^2, \qquad (89.1)$$

$$2\,V = \mu_1^2 q_1^2 + \mu_2^2 q_2^2 + \cdots + \mu_m^2 q_m^2; \qquad (89.2)$$

and, on the assumption that there are no dissipation or external forces acting on the system, our Lagrangian equations (78.3) take the form

$$\ddot{q}_s = -\,\mu_s^2 q_s, \quad (s = 1, 2, \cdots, m)$$

of which the general integral is

$$q_s = C_s \cos(\mu_s t - \delta_s), \quad (s = 1, 2, \cdots, m) \qquad (89.3)$$

with the proper number $2\,m$ of arbitrary constants. In order to determine these constants it is necessary to know the initial phase of the system, which amounts to a knowledge of the initial values of the q's and their time derivatives $\dot{q}$. Since, however, the various coördinates q are entirely separated from each other in the solution (89.3), it is clear that the values of any particular amplitude C_s and the corresponding δ_s are completely determined by a knowledge of the initial values of the single coördinate q_s and its time derivative $\dot{q}_s$. The subsequent values of this coördinate are unaffected by any change made in the initial values of the other coördinates or of their velocities, and for this reason the *normal* vibrations (89.3) of the system are said to take place independently of each other.

Upon substituting the values (89.3) in the expressions (89.1) and (89.2) for the kinetic and potential energies respectively, we get

$$2\,T = C_1^2\mu_1^2 \sin^2(\mu_1 t - \delta_1) + \cdots + C_m^2\mu_m^2 \sin^2(\mu_m t - \delta_m), \quad (89.4)$$

$$2\,V = C_1^2\mu_1^2 \cos^2(\mu_1 t - \delta_1) + \cdots + C_m^2\mu_m^2 \cos^2(\mu_m t - \delta_m). \quad (89.5)$$

If we take the time average of the $\sin^2$ functions in (89.4), we get, for example,

$$\frac{1}{t}\int_0^t \sin^2(\mu_1 t - \delta_1)dt = \frac{1}{2} - \frac{\sin 2(\mu_1 t - \delta_1)}{4t}.$$

This tends to $\frac{1}{2}$ as t tends to infinity. Hence the time average of the kinetic energy of the system over a long period is

$$\overline{T} = \tfrac{1}{4}(C_1{}^2\mu_1{}^2 + \cdots + C_m{}^2\mu_m{}^2), \tag{89.6}$$

and the time average of the potential energy over a long period is the same as this. In other words, the energy of the system, when averaged with respect to time over a long period, is half kinetic and half potential.

It may be observed that the characteristic numbers λ_r have now a physical interpretation in the fact that the period of the rth normal vibration of the system is $\frac{2\pi}{\lambda_r}\cdot$ We have, then, from the mathematical treatment in § 88, Rayleigh's theorem:

When a single constraint is imposed on a system performing small oscillations about a position of stable equilibrium, the periods of the new normal vibrations separate the original periods of the normal vibrations.

If more than one constraint is put on the system, we have the more general result given in (88.1).

90. Forced vibrations about a position of stable equilibrium. We shall now consider the case where, in addition to the forces derivable from the potential-energy function V, external forces are acting which depend only on the time and not at all on the position or velocities of the system. Using normal coördinates q, and denoting the virtual work of the external forces by $Q_1\,\delta q_1 + Q_2\,\delta q_2 + \cdots + Q_m\,\delta q_m$, the Lagrangian equations governing the motion are

$$\ddot{q}_s = -\mu_s{}^2 q_s + Q_s, \quad (s = 1, 2, \cdots, m) \tag{90.1}$$

where Q_s is a function of the time alone. The variables q are entirely separated from one another in the equations (90.1), and so the discussion given in § 40 is applicable. The function Q_s of t being expanded in a Fourier series of sines and cosines of multiples of t, we may consider each term of this expansion separately. We have, then, to solve an equation of the type

$$\ddot{q}_s + \mu_s^2 q_s = A_n \cos nt,$$

of which a particular solution is

$$q_s = \frac{A_n}{\mu_s^2 - n^2} \cos nt,$$

provided that $n \neq \mu_s$. If $n = \mu_s$, the particular solution is $q_s = \left(\frac{A_n t}{2n}\right) \sin nt$. The general solution is obtained by adding the particular solution to the free vibrations (89.3). When the period of an exciting force is nearly equal to one of the free periods, we have the phenomenon of resonance (see § 40).

EXAMPLE

The double pendulum. Here we have two masses m_1 and m_2, the first of which is attached to a fixed point by a light, inextensible string of length l_1, the second being attached by a similar thread, of length l_2, to the mass m_2. The position of the system at any instant is fixed by a knowledge of the two angles θ_1 and θ_2 of inclination of the threads to the vertical. The kinetic energy of the two masses is

$$T = \frac{m_1 l_1^2 \dot{\theta}_1^2}{2} + \frac{m_2}{2} [l_1^2 \dot{\theta}_1^2 + 2\, l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2].$$

In the position of stable equilibrium, $\theta_1 = 0 = \theta_2$; and since we are regarding θ_1 and θ_2 and their velocities as infinitesimals, we may substitute for $\cos(\theta_1 - \theta_2)$, in the expression for T, its value, unity, in the position of stable equilibrium. The potential energy is

$$V = m_1 g l_1 (1 - \cos \theta_1) + m_2 g (l_1 + l_2 - l_1 \cos \theta_1 - l_2 \cos \theta_2),$$

where the additive constant has been so adjusted that V is zero in the position of equilibrium. Neglecting powers of θ_1 and θ_2 higher than the second, we have

$$V = \tfrac{1}{2}(m_1 + m_2) g l_1 \theta_1^2 + \tfrac{1}{2}\, m_2 g l_2 \theta_2^2.$$

The coördinates θ_1 and θ_2 are not normal, since a product term occurs in the expression for T. Writing $\theta_1 = A_1 e^{\lambda t}$, and $\theta_2 = A_2 e^{\lambda t}$, we get two equations, as in (86.1). Equating the two expressions for the ratio $\frac{A_1}{A_2}$ obtained from these equations, we get the fourth-degree equation in λ,

$$m_1 l_1 l_2 \lambda^4 + \lambda^2 (m_1 + m_2) g (l_1 + l_2) + g^2 (m_1 + m_2) = 0, \quad (90.2)$$

which may be treated as a quadratic equation for λ^2. It has two real, negative roots which we shall denote by $-\mu_1^2$ and $-\mu_2^2$. The general solution of the problem is, then,

$$\theta_1 = C_{11} \cos (\mu_1 t - \delta_1) + C_{12} \cos (\mu_2 t - \delta_2),$$
$$\theta_2 = C_{21} \cos (\mu_1 t - \delta_1) + C_{22} \cos (\mu_2 t - \delta_2),$$

where the constants C_{11}, C_{12}, δ_1, δ_2 are arbitrary, whereas the remaining constants C_{21} and C_{22} are known in terms of C_{11} and C_{12}. In fact, $C_{21} = \frac{l_1\mu_1^2 C_{11}}{g - l_2\mu_1^2}$, $C_{22} = \frac{l_1\mu_2^2 C_{12}}{g - l_2\mu_2^2}$.

The special case where the two pendulums are alike may well be treated in more detail. Here $m_1 = m_2$, $l_1 = l_2$; and we may drop their distinguishing subscripts. On solving the quadratic (90.2) in this particular case, we find

$$\mu_1^2 = \frac{g}{l}(2 + \sqrt{2}), \quad \mu_2^2 = \frac{g}{l}(2 - \sqrt{2}).$$

If we denote by μ_0 the frequency $\sqrt{\frac{g}{l}}$ of a simple pendulum of length l, the two frequencies of the double pendulum are, approximately,

$$\mu_1 = 1.85\ \mu_0, \quad \mu_2 = .76\ \mu_0.$$

We leave it to the student to show that when the lengths of the two pendulums are equal, without the masses being necessarily equal,

$$\left.\begin{matrix}\mu_1^2 \\ \mu_2^2\end{matrix}\right\} = \frac{m_1 + m_2}{m_1}\mu_0^2\left(1 \pm \sqrt{1 - \frac{m_1}{m_1 + m_2}}\right).$$

BIBLIOGRAPHY

The subject of small vibrations about a position of stable equilibrium is discussed in all the standard texts; but the following work is to be particularly recommended:

Routh, E. J. Advanced Rigid Dynamics. London, 1905.

In this treatise the connected question of the oscillations of a dynamic system about a state of steady motion is also discussed in detail.

Courant, R., and Hilbert, D. Methoden der mathematischen Physik. Berlin, 1924.

This recently published book is also highly recommended. Modern mathematical methods are freely used in this text, which will be found to complement the older work of Routh.

Rayleigh. Theory of Sound. New York, 1926.

This is particularly recommended for the physical insight which it gives into the mathematics of the subject.

EXERCISES

1. Two particles connected by a rigid rod move on a smooth vertical circle. Find the time of a small oscillation.

2. A uniform rod AB is suspended from a fixed point O by a short rod OC which is attached to it at right angles at its middle point. Equal weights are suspended from A and B by strings of equal lengths, the whole system forming a somewhat sluggish balance. If one weight is drawn slightly aside from the vertical and allowed to oscillate, the system starting from rest, find the subsequent motion.

3. A uniform bar is suspended by two equal parallel strings from two points in the same horizontal line. Find the period of the small oscillations of the bar when it is turned through a small angle about the vertical through its middle point and is let go.

4. A homogeneous hemisphere performs small oscillations on a perfectly rough horizontal plane. Find the motion.

5. A homogeneous sphere makes small oscillations inside a fixed sphere so that its center moves in a vertical plane. If the roughness is sufficient to prevent all sliding, prove that the length of the equivalent pendulum is seven fifths of the difference of the radii. If the spheres were smooth, the length of the equivalent pendulum would be equal to the difference of the radii.

6. Discuss, in the manner of the example on page 302, the complex torsional pendulum, where two rods are supported by a vertical fiber connecting their centers of mass, so that each rod vibrates in a horizontal plane. The moments of inertia of the rods about the vertical axis are given.

CHAPTER XIII

THE PRINCIPLE OF LEAST ACTION

91. An absolute description of the motion of a dynamic system. Although Lagrange's equations (78.3) are general, in the sense that the q's may be any coördinates (independent) which serve to give the position of the system, they have the disadvantage — at any rate for a theoretical discussion — that they are *second-order* differential equations and not first-order equations. This defect was remedied by the introduction of the momenta p and the derivation of the canonical equations (80.7) and (80.13). However, these $2\,m$ equations of the first order in which the dependent variables are the phase coördinates (q, p), and the independent variable is the time, have now lost some of the generality which was such an important feature of the Lagrangian equations (78.3); for the phase coördinates (q, p) may not be *any* coördinates sufficient to describe the phase of the system. The q's are still arbitrary position coördinates, but the momenta are not at our free disposal. They are definitely defined by the equations (79.1). It is our purpose in the present paragraph to state the laws governing the motion so that the differential equations may be written down as soon as we have any coördinates whatsoever specifying the phase of the system. These coördinates may be *any* $2\,m$ distinct functions (s, r) of the original phase coördinates (q, p), and they may also involve the time variable explicitly. Thus

$$s_n = s_n(q, p, t), \quad r_n = r_n(q, p, t). \quad (n = 1, 2, \cdots, m) \quad (91.1)$$

Since these functions of the (q, p) are supposed to be distinct, we may solve equations (91.1) for the (q, p) in terms of the (s, r), the solutions involving the time variable explicitly if equations (91.1) involve t explicitly. The argument followed will not make any use of the fact that t is chosen as the independent variable in the canonical equations, and that the phase coördinates are to be expressed in terms of t. All the *state* coördinates

(q, p, t) (see § 79) may be regarded as on an equal footing, and our problem is to state the laws governing the motion of the dynamic system so that the differential equations may be immediately written down as soon as we have any $2m+1$ coördinates which serve to tell the state of the system, that is, its phase-time. These coördinates may be any $2m+1$ distinct functions (s, r, τ) of the original state coördinates (q, p, t). Thus

$$\begin{aligned} &s_n = s_n(q, p, t), \quad r_n = r_n(q, p, t), \quad (n = 1, 2, \cdots, m) \\ &\tau = \tau(q, p, t). \end{aligned} \tag{91.2}$$

Since these functions are supposed to be distinct, we may solve them and express (q, p, t) as functions of (s, r, τ); so that if the latter $2m+1$ numbers are known, the state of the dynamic system is also known. The desired statement of the laws of motion of the dynamic system may be called an absolute statement, since it has no reference to any particular system of state coördinates. There will exist no *privileged* systems of coördinates such as exist when we use the canonical equations, where the m momenta coördinates p must be related to the position coördinates q by means of equations (79.1). In order to give the absolute statement of the laws of motion of the system, it will be convenient to use the so-called geometric language described in § 79, no idea of measurement being, however, involved. In this language a point in a state space of $2m+1$ dimensions represents a particular state of the system; and as the state changes we obtain in the state space a curve which represents the life history of the system. Such a curve may be called a *path* of the system; and the canonical equations (80.7) or (80.13), as the case may be, are the differential equations of these paths, the variables being the original state coördinates. Since there is only one solution of a system of differential equations of the first order satisfying given initial conditions, there can be only one path of the system passing through any point of the state space. The particular property of these paths which we wish to prove is that they are the extremal curves for a certain line integral known as the *action integral.* By this is meant that if we take two points lying on the same path and evaluate the action integral along that portion of the path lying between these two points, we shall obtain a result which is stationary when compared with the value of the action integral along neighboring curves joining the same two points. Since two paths never intersect, these neighboring curves cannot themselves be paths. The action along any curve, whether it is a path or not, is, since it is a line integral, completely determined when we know the curve; and so this action may be called a function of the curve (see § 13, *a*). If we consider all curves joining the two points which lie on the same path, our theorem is that the action has a stationary value for the actual path itself. In other words, as we vary the curve from the path the rate of change of the action integral is zero. In terms of the original state coördinates, the line integral which we call the action integral is

$$A = \int p_1\,dq_1 + p_2\,dq_2 + \cdots + p_m\,dq_m - H\,dt. \qquad (91.3)$$

(We are limiting ourselves to the cases where the canonical equations may be put in the form (80.7).) The linear differential form under the sign of integration may be called the action integrand. It will be observed that the differentials of the momenta p do not enter the action integrand (or, if we prefer, their coefficients are zero), and that the coefficient of the differential of each position coördinate q_s is the corresponding momentum coördinate p_s. The coefficient of the one remaining state coördinate, the time, is the negative of the Hamiltonian function H. In order to prove the extremal character of the paths with reference to the action line integral, it will be convenient to introduce, for the moment, a more symmetrical notation.

Let us consider a space of n dimensions, the coördinates of any point being denoted by $(x_1, x_2, \cdots, x_n)$. Let us seek the curves which are extremal curves for a given line integral

$$I = \int C_1\,dx_1 + C_2\,dx_2 + \cdots + C_n\,dx_n, \qquad (91.4)$$

where the coefficients C_s are known functions of the coördinates x. When we want to return to the previous rather unsymmetrical notation, we shall observe that

$$n = 2m+1,\quad x_s = q_s,\quad x_{m+s} = p_s,\quad x_{2m+1} = t,\quad (s = 1, 2, \cdots, m) \qquad (91.5)$$

$$C_s = p_s,\quad C_{m+s} = 0,\quad C_{2m+1} = -H.\quad (s = 1, 2, \cdots, m) \qquad (91.6)$$

It is important to notice that the number of dimensions $n = 2m+1$ of the state space to which we shall apply our results is *odd*. The curve along which the integral (91.4) is to be evaluated may be supposed to be given by certain equations

$$x_s = x_s(\theta), \quad (s = 1, 2, \cdots, n) \tag{91.7}$$

where θ is the independent variable parameter whose rôle is to individualize the various points on the curve of integration. When the curve (91.7) and the limits θ_0 and θ_1 of integration are given, I, as defined in (91.4), is a fixed number and not a variable. In order to enable I to vary, we shall have to introduce another independent variable or parameter ϕ, whose rôle is to change, when the value assigned to it is changed, the curve of integration, this change of the curve of integration involving, in general, a change of the end points. Our equations (91.7) will, therefore, be replaced by equations of the type

$$x_s = x_s(\theta, \phi), \quad (s = 1, 2, \cdots, n) \tag{91.8}$$

where ϕ remains constant along any curve of integration. The reason for its appearance in equations (91.8) is that we are thereby enabled to vary the curve of integration and thus obtain variable values of I. The variables θ and ϕ are quite independent, the rôle of the first being to individualize points on the curve along which the second remains constant, and the rôle of the second to individualize the curves of integration. The end points of these curves of integration may be given by equations

$$x_s^0 = x_s(\theta^0, \phi), \quad x_s^1 = x_s(\theta^1, \phi), \quad (s = 1, 2, \cdots, n) \tag{91.9}$$

where (θ^0, θ^1) are certain constants independent of ϕ. If the end points of all the curves of integration are the same, the variable ϕ will not actually enter the expressions (91.9). The really important thing to grasp is that although the end points of the curves of integration may vary, the limits of the integral (91.4), when this integral is expressed explicitly as an ordinary definite integral with the single variable of integration θ, do not vary from curve to curve, that is, as ϕ is changed in value. The change, if any, in the end points is brought about through the explicit appearance of ϕ in the expressions (91.9).

Now since the x's, as given in (91.8), are functions of two independent variables, we have two partial differentials to distinguish. The first is the differential

$$dx_s = \frac{\partial x_s}{\partial \theta}\, d\theta, \quad (s = 1, 2, \cdots, n) \tag{91.10}$$

which occurs in (91.4) and is calculated on the basis that ϕ is kept constant. The other partial differential may be denoted, in order to distinguish it from this one, by δx_s, where

$$\delta x_s = \frac{\partial x_s}{\partial \phi}\, d\phi. \tag{91.11}$$

It is calculated on the basis that θ is kept constant. Now the integral I, as defined by (91.4), is a function of the single variable ϕ, which enters it through the x's, of which the coefficients C are supposed to be functions, and also through the differentials of these x's, which are multiplied by the coefficients C. These are the only places where ϕ enters the integral I, the limits of this integral being, as we have just remarked, independent of ϕ. We find, therefore, the differential of I by merely integrating with respect to θ the partial differential of the integrand with respect to ϕ; that is,

$$\delta I = \int \sum_r (C_r \, \delta dx_r + \delta C_r \, dx_r). \tag{91.12}$$

Now

$$\delta dx_r = \frac{\partial}{\partial \phi}\left(\frac{\partial x_r}{\partial \theta} \, d\theta\right) d\phi = \frac{\partial^2 x_r}{\partial \phi \, \partial \theta} \, d\theta \, d\phi,$$

and, since the order of differentiation in finding the second derivative may be inverted, this is equivalent to $d\delta x_r$. Hence the first group of terms of the summation in the integrand of (91.12) may be integrated by parts, using

$$\int C_r \, d\delta x_r = C_r(\delta x_r)_0^1 - \int dC_r \, \delta x_r.$$

On developing the two partial differentials of C_r, that is,

$$dC_r = \sum_s \frac{\partial C_r}{\partial x_s} \, dx_s, \quad \delta C_r = \sum_s \frac{\partial C_r}{\partial x_s} \, \delta x_s, \tag{91.13}$$

we have the result

$$\delta I = \left(\sum_r C_r \, \delta x_r\right)_0^1 + \int \sum_{r,\, s} \left(\frac{\partial C_r}{\partial x_s} \, dx_r \, \delta x_s - \frac{\partial C_r}{\partial x_s} \, \delta x_r \, dx_s\right).$$

On collecting the terms under the sign of integration, we have

$$\delta I = \left(\sum_r C_r \, \delta x_r\right)_0^1 + \int \sum_{r,\, s} C_{rs} \, \delta x_r \, dx_s, \tag{91.14}$$

where we have written, for the sake of brevity,

$$C_{rs} = \frac{\partial C_s}{\partial x_r} - \frac{\partial C_r}{\partial x_s}. \quad \left(\begin{Bmatrix} r \\ s \end{Bmatrix} = 1, 2, \cdots, n\right) \tag{91.15}$$

If we now make the stipulation that the end points of the curves of integration shall all be the same, the part of δI in (91.14) outside the sign of integration will vanish, since every δx_r will be zero, at both the upper and lower limits. We have, then, in this special case the simpler expression

$$\delta I = \int \sum_{r,\, s} C_{rs} \, \delta x_r \, dx_s, \tag{91.16}$$

where the coefficients C_{rs} are defined in terms of the given coefficients of the line integral I by means of equations (91.15). In order that δI

may be zero for arbitrary values of the partial differentials δx_r (that is, the differentials with respect to the parameter ϕ which changes the curve of integration), it is necessary and sufficient that the coefficients of each δx_r in the integrand of (91.16) should separately vanish; and we thus obtain the equations, n in number,

$$\sum_s C_{rs}\, dx_s = 0. \quad (r = 1, 2, \cdots, n) \tag{91.17}$$

Since the dx's are the partial differentials of x with respect to θ, that is, along the curve of integration, these equations may be regarded as the differential equations of the extremal curves for the given line integral I. Since we have exactly n linear homogeneous equations for the n differentials dx_s, we know that there will be no solution other than the trivial one $dx_s = 0$ $(s = 1, 2, \cdots, n)$ unless the determinant of the coefficients is zero. This means that the given integral will not always have extremal curves. It will be observed, however, from the equations of definition (91.15), that $C_{sr} = - C_{rs}$, so that an interchange of rows and columns in the determinant of the coefficients will multiply the determinant by $(-1)^n$; but this interchange of rows and columns cannot affect the value of the determinant. Hence the determinant of the coefficients of equations (91.17) will be zero, no matter what the given line integral I is, provided n is an odd number. In the particular application that we have in view, this is the case, since $n = 2m + 1$ (see 91.5); and we are therefore assured of the existence of extremal curves for any line integral I.

We find, then, that the extremal curves for the line integral

$$I = \int \sum_r C_r\, dx_r$$

are the integral curves or solutions of the set of n linear homogeneous differential equations

$$\sum_s C_{rs}\, dx_s = 0, \quad (r = 1, 2, \cdots, n)$$

where the coefficients C_{rs} are defined by the equations

$$C_{rs} = \frac{\partial C_s}{\partial x_r} - \frac{\partial C_r}{\partial x_s}$$

and are therefore alternating, that is, $C_{sr} = - C_{rs}$. These n differential equations are not independent, so that one of them will be automatically satisfied by a solution of the remaining $n - 1$ equations. Let us now apply this result to the action integral

$$A = \int \sum_r p_r\, dq_r - H\, dt.$$

Upon substituting the values given in (91.5) and (91.6), we have the following results:

$$\begin{aligned}
C_{rs} &= 0, \qquad (r,\, s = 1,\, 2,\, \cdots,\, m)\\
C_{r,\, m+r} &= -1, \qquad (r = 1,\, 2,\, \cdots,\, m)\\
C_{r,\, m+s} &= 0, \qquad (r,\, s = 1,\, 2,\, \cdots,\, m\,;\ r \neq s)\\
C_{r,\, 2m+1} &= -\frac{\partial H}{\partial q_r}, \quad (r = 1,\, 2,\, \cdots,\, m)\\
C_{m+r,,\, 2m+1} &= -\frac{\partial H}{\partial p_r}. \quad (r = 1,\, 2,\, \cdots,\, m)
\end{aligned}$$

In deducing these results it is, of course, to be remembered that the partial differentiations involved in the definition (91.15) of the expressions C_{rs} are made on the basis that all the variables x, here $(q,\, p,\, t)$, are independent, so that the partial derivative of any one of them with respect to any other is zero. The differential equations of the extremal curves for the action integral are accordingly, by (91.17)

$$\begin{gathered}
-\, dp_r - \frac{\partial H}{\partial q_r}\, dt = 0, \quad (r = 1,\, 2,\, \cdots,\, m)\\
dq_r - \frac{\partial H}{\partial p_r}\, dt = 0, \quad (r = 1,\, 2,\, \cdots,\, m)\\
\sum_r \left(\frac{\partial H}{\partial q_r}\, dq_r + \frac{\partial H}{\partial p_r}\, dp_r\right) = 0.
\end{gathered}$$

Our general theory shows us that these $2\,m + 1$ equations are not independent, and it is, in fact, apparent that the last one is a linear combination of the other $2\,m$ equations. These $2\,m$ equations are, however, exactly the canonical equations (80.7) governing the motion of the dynamic system; and so we have the fundamental result

Any dynamic system possessing a Hamiltonian function H will move so that its representative point in the $2\,m + 1$ dimensional state space will trace out an extremal curve of the action integral

$$A = \int p_1\, dq_1 + p_2\, dq_2 + \cdots + p_m\, dq_m - H\, dt.$$

By this is meant that when we compare the action along the actual path followed with the action along all other curves in the state space joining the initial and final states of the system, the action has a stationary value at the actual path. If now we wish to write down the dynamic equations when any state coördinates (s, r, τ) whatsoever are used, we have merely to write out the action integral in these coördinates. This is done by substituting for each p its expression in terms of the new state coördinates and by writing for each dq its value

$$\sum_k \left(\frac{\partial q}{\partial s_k} ds_k + \frac{\partial q}{\partial r_k} dr_k\right) + \frac{\partial q}{\partial \tau} d\tau,$$

and similarly for H and dt. Let the new form of the action integral be

$$A = \int \sum_k (S_k\, ds_k + R_k\, dr_k) + K\, d\tau.$$

Then the dynamic equations are the equations (91.17) of the extremals of this integral.

92. The Hamilton-Jacobi partial-differential equation. We have seen in the preceding section that the partial differential of the action integral, with respect to the parameter which serves to vary the curve of integration, is zero when the curve of integration is a path of the dynamic system, *provided the initial and final states of the system are prescribed.* If the initial and final states may be varied, this partial differential, instead of being zero, has the value

$$\delta A = \Big(\sum_k p_k\, \delta q_k - H\, \delta t\Big)_0^1 \tag{92.1}$$

when the curve of integration is an actual path (see (91.14)). Now the action along a path of the system is completely determined by the initial and final states, since the initial state determines definitely the path. Only one of the final state coördinates is at our choice when the initial state is given, the other final state coördinates being determined by this one. In other words, the action integral is a function of $2m + 2$ independent variables. The equation (92.1) suggests that a good choice of independent variables in terms of which to express the action integral might be the initial position and time coördinates (q^0, t^0) and the final position and time coördinates (q, t). If this could be done, we should have, as an equivalent way of stating equation (92.1), the set of equations

$$\begin{gathered}\frac{\partial A}{\partial q_k} = p_k, \quad \frac{\partial A}{\partial q_k{}^0} = -\,p_k{}^0, \quad (k = 1, 2, \cdots, m) \\ \frac{\partial A}{\partial t} = -\,H, \quad \frac{\partial A}{\partial t^0} = H^0.\end{gathered} \tag{92.2}$$

The function H is given to us as a function of the state coördinates (q, p, t), and we may eliminate the momenta p from it by means of the equations (92.2). In this way we get

$$\frac{\partial A}{\partial t} = - H\left(q, \frac{\partial A}{\partial q}, t\right), \tag{92.3}$$

where we understand by the right-hand side of the equation the result obtained when each p_s in the given function H is replaced by the corresponding partial derivative $\frac{\partial A}{\partial q_s}$. Equation (92.3) is a partial-differential equation of the first order and second degree for the single function A of $m + 1$ independent variables (q, t). Now a partial-differential equation may have many solutions, and the action integral A is distinguished from the many other solutions of (92.3) by the fact that it involves certain arbitrary constants in such a way that the partial derivatives of A with respect to these constants are exactly the initial momenta $p_k{}^0$, the arbitrary constants themselves being the initial position coördinates. If we were able to determine this particular solution of equation (92.3), or, in other words, if we were able to express the action along a path as a function of the initial position and time coördinates (q^0, t^0) and the final position and time coördinates (q, t), the equations

$$\frac{\partial A}{\partial q_k} = p_k, \quad \frac{\partial A}{\partial q_k{}^0} = - p_k{}^0, \quad (k = 1, 2, \cdots, m)$$

of (92.2) would enable us by algebraic methods, involving no integration whatever, to express all but one of the final state coördinates in terms of this remaining one and the initial state coördinates. In effect, we should have solved the problem of the motion of the system; for we should be able to tell the phase at any time from a knowledge of the initial phase. However, the problem of determining that special solution of the partial-differential equation (92.3) whose partial derivatives, with respect to the arbitrary constants it contains, are exactly the initial momenta when the constants are given the values of the initial position coördinates, baffled Hamilton; and his remark that the action integral, when expressed in terms of the initial and final states of the system, satisfies the first-order partial-differential equation (92.3) remained unused until Jacobi showed that it is entirely unnecessary to limit ourselves to the action integral in order to determine the motion of the system. Any solution of the partial-differential equation (92.3) involving the right number of arbitrary constants, provided these enter the solution in the proper way, will serve the purpose just as well as the action integral, which is a special solution of this equation. For this reason the partial-differential equation (92.3) is known as the Hamilton-Jacobi partial-differential equation, the situation being that Hamilton was the discoverer of the equation, whereas it remained to Jacobi to make the discovery useful.

In order to prove Jacobi's contention it will be convenient to return to the absolute form of statement of the laws governing the motion of the dynamic system which has been given in § 91. Let us suppose that we are able to find a transformation of state coördinates (q, p, t) to (s, r, τ), say, such that the action integrand

$$\sum_k p_k\, dq_k - H\, dt$$

transforms into one of the same form,

$$\sum_k r_k\, ds_k - J\, d\tau.$$

It then follows that the dynamic differential equations, which express the fact that the paths of the system in the state space are extremal curves for the action integral, will have exactly the same form in the new state coördinates as in the old; for the action integrand has exactly the same form. They will be, therefore,

$$\frac{ds_k}{d\tau} = \frac{\partial J}{\partial r_k}, \quad \frac{dr_k}{d\tau} = -\frac{\partial J}{\partial s_k}. \quad (k = 1, 2, \cdots, m) \tag{92.4}$$

It is evident, furthermore, that (92.4) will still be the dynamic equations even when the form of the action integrand is not preserved, provided that the change in form amounts only to the addition of an exact differential; that is, when the new form of the action integrand is

$$\sum_k r_k\, ds_k - J\, d\tau + dS, \tag{92.5}$$

where S is any function of the state coördinates (s, r, τ). For the addition of this exact differential to the previous form of the action integrand changes the action integral by a function of the end points only, and not of the curve of integration. As the end points are supposed to be held fixed in finding the extremals, the extremal curves for the action integrand (92.5) must be the same as the extremal curves for the first part,

$$\int\Big(\sum_k r_k\,ds_k - J\,d\tau\Big),$$

of the action integral. We have, then, the result that the canonical equations (80.7) preserve their form under any transformation of state coördinates (q, p, t) to (s, r, τ) which sends the action integrand $\sum_k p_k\,dq_k - H\,dt$ into the form (92.5). By analogy with similar transformations in the geometric theory of surfaces, such transformations are known as contact transformations. We shall now try to find those particular contact transformations which give a form (92.5) of the action integral in which the differential of the time variable is missing, that is, $J = 0$. The advantage of such special contact transformations is that the dynamic differential equations become immediately integrable. They are, in fact,

$$\frac{ds_k}{d\tau} = 0, \quad \frac{dr_k}{d\tau} = 0, \quad (k = 1, 2, \cdots, m) \tag{92.6}$$

so that all the coördinates (s, r) are constant along the path. The equations of the paths in terms of the original state coördinates (q, p, t) follow at once by making the $2\,m$ symbols (s, r) each equal to a constant in the equations of transformation from (q, p, t) to (s, r, τ), thus expressing the $2\,m + 1$ state coördinates (q, p, t) in terms of the single parameter τ and the $2\,m$ arbitrary constants (s, r).

It remains to show how to find a reversible transformation of state coördinates which will send the action integral into the form $\sum_k r_k\,ds_k + dS$. We have the equation

$$\sum_k p_k\,dq_k - H\,dt = \sum_k r_k\,ds_k + dS, \tag{92.7}$$

which is very similar to (92.1). On supposing S to be expressed as a function of the variables (q, s, t, τ) instead of the variables (s, r, t, τ), we have the equations

$$\begin{gathered} \frac{\partial S}{\partial q_k} = p_k, \quad \frac{\partial S}{\partial s_k} = -\,r_k, \quad (k = 1, 2, \cdots, m) \\ \frac{\partial S}{\partial t} = -\,H, \quad \frac{\partial S}{\partial \tau} = 0. \end{gathered} \tag{92.8}$$

Upon eliminating the p's from H by means of these equations, we see that the function S of (q, t) must satisfy the partial-differential equation of the first order,

$$\frac{\partial S}{\partial t} = -\,H\Big(q, \frac{\partial S}{\partial q}, t\Big), \tag{92.9}$$

where in the partial differentiations the other variables which appear in S are regarded as constants. Equation (92.9) is exactly the equation, (92.3), which Hamilton showed was satisfied by the action integral when this was expressed in terms of the initial and final state coördinates. It has one dependent variable S and $m + 1$ independent variables (q, t). What is known as a complete integral of such a partial-differential equation is one involving as many arbitrary constants as there are independent variables. *These constants enter the integral in such a way that the initial values of the dependent variable and its partial derivatives with respect to the independent variables may be given initially any values whatsoever consistent with the partial-differential equation.*

In the particular case of (92.9) — which is of a special type, inasmuch as the dependent variable does not enter the equation explicitly but only through its partial derivatives — one of the arbitrary constants is plainly an additive constant which may be used to fix the initial value of S. As we shall be interested only in the partial derivatives of S, and not at all in the actual values assumed by S itself, we shall not trouble to use this additive constant. We shall try to find, then, a complete integral of (92.9) — that is, an integral involving m arbitrary constants, none of which is purely additive, in such a way that the m partial derivatives of S with respect to the position coördinates q may be given arbitrary initial values. The initial value of the remaining partial derivative $\frac{\partial S}{\partial t}$ is then determined by the differential equation (92.9) itself. If we denote the initial values assigned to the partial derivatives $\frac{\partial S}{\partial q_k}$ by $p_k{}^0$ $(k = 1, 2, \cdots, m)$, our condition as to the way the constants must enter the solution of (92.9) amounts to the fact that we must be able to solve the m equations

$$\left(\frac{\partial S}{\partial q_k}\right)^0 = p_k{}^0 \quad (k = 1, 2, \cdots, m) \tag{92.10}$$

for the s_k's in terms of $(q_k{}^0, p_k{}^0, t)$. Mathematically expressed, the criterion by which we know that this solution is possible in any particular case is that the Jacobian determinant of the m partial derivatives $\frac{\partial S}{\partial q}$ with respect to the m arbitrary constants s must not vanish identically. The element in the ith row and jth column of this Jacobian determinant is the mixed derivative $\frac{\partial^2 S}{\partial s_i \partial q_j}$. As the order of differentiation is immaterial, this shows us that if the equations (92.10) can be solved for s_k, the new set of equations

$$\frac{\partial S}{\partial s_k} = -r_k \quad (k = 1, 2, \cdots, m) \tag{92.11}$$

can be solved for the q's in terms of (s_k, r_k, t).

We are now able to state the fundamental result:

A knowledge of a complete integral of the Hamilton-Jacobi partial-differential equation (92.9) *determines a reversible transformation of state coördinates from* (q, p, t) *to* (s, r, t) *by means of the two sets of* m *equations*

$$\frac{\partial S}{\partial q_k} = p_k, \quad \frac{\partial S}{\partial s_k} = -r_k, \tag{92.12}$$

to which we may add, if we like, the identical relation $t = t$, *to show that we are not changing the time coördinate.*

This transformation of state coördinates is a contact transformation of the special type characterized by the fact that the differential of the time variable is missing from the action integrand when this is expressed in the form (92.5). Hence the laws governing the motion of the system are simply

$$\frac{ds_k}{dt} = 0, \quad \frac{dr_k}{dt} = 0; \quad (k = 1, 2, \cdots, m)$$

so that, in terms of the new state coördinates, the paths are

$$s_k = \text{constant}, \quad r_k = \text{constant}. \quad (k = 1, 2, \cdots, m)$$

In terms of the original state coördinates (q, p, t) the paths representing the motion of the system are found by solving the reversible transformation (92.12) for the phase coördinates (q, p) in terms of the constants (s, r) and the time variable t. These constants can be so chosen that for any initial time t^0 the system may be given any phase (q^0, p^0) we wish. After the determination of a complete integral of the Hamilton-Jacobi equation, no further integrations are necessary to write down the equations of the paths. All we have to do is to solve equations (92.12), which are "finite" and not differential equations. In general, it will not be necessary to give the explicit form of the equations of the paths, and these may be left in the implicit form (92.12).

The truth of the statements just made follows at once from the previous argument. The transformation is reversible, since the second set of equations, that is, $\frac{\partial S}{\partial s_k} = -r_k$ $(k = 1, 2, \cdots, m)$, can be solved for the q's in terms of (s, r, t), and the substitution of these results in the other set, that is, $\frac{\partial S}{\partial q_k} = p_k$ $(k = 1, 2, \cdots, m)$, then gives us the p's directly in terms of (s, r, t). In a similar way, on interchanging the rôles played by the two sets of m equations, we can express first the s's and then the r's in terms of (q, p, t). That the transformation is a contact transformation of the special type desired is evident since, by (92.12) and (92.9),

$$dS = \sum_k \frac{\partial S}{\partial q_k}\, dq_k + \sum_k \frac{\partial S}{\partial s_k}\, ds_k + \frac{\partial S}{\partial t}\, dt = \sum_k (p_k\, dq_k - r_k\, ds_k) - H\, dt;$$

so that
$$\sum_k p_k\, dq_k - H\, dt = \sum_k r_k\, ds_k + dS.$$

93. Reduction of the Hamilton-Jacobi equation when some of the coördinates are cyclic. We have already seen that when a position or time coördinate does not enter the Hamiltonian function H, considerable simplification of the problem results. Let us see how this appears in the finding of a complete integral of the partial-differential equation (92.9). First we may consider the case where the time variable does not appear explicitly in H. We have, then, the energy integral $H =$ constant, (80.8), and we shall denote this constant by h. It is possible, in this case, to find a complete integral of (92.9) involving the time linearly. In fact, if we make a change of dependent variable from S to W, say, by means of the substitution

$$S = -ht + W(q), \tag{93.1}$$

where the new dependent variable W does not involve t, our equation (92.9) becomes

$$h = H\left(q, \frac{\partial W}{\partial q}\right), \tag{93.2}$$

since the partial derivative of S with respect to t is $-h$, and since its partial derivative with respect to any position coördinate q is the same as the partial derivative of W with respect to that coördinate. The new partial-differential equation (93.2) has one dependent and m independent variables, so that there is one less independent variable than in (92.9). The equation will, then, be, in general, much easier to deal with. A complete integral of (93.2) will be a function, W, of the m variables q, satisfying the equation and involving $m - 1$ arbitrary constants, none of which is merely additive, in such a way that the m partial derivatives of W with respect to the q's may, for any initial values q^0 of these variables, be given any values whatsoever satisfying equation (93.2). Let us suppose we have found such a complete integral, and let us denote the $m-1$ arbitrary constants by $(s_2, s_3, \cdots, s_m)$. Besides these, W will involve the constant h, which takes the place of the missing constant s_1. Then S, as defined by (93.1), will be a complete integral of (92.9). It involves, in fact, the proper number m of arbitrary constants $(h, s_2, \cdots, s_m)$. Since each $\frac{\partial S}{\partial q}$ is equal to the corresponding $\frac{\partial W}{\partial q}$, the last $m-1$ of these can be so chosen that the $\frac{\partial S}{\partial q}$'s may be given any initial values, the constant h being then determined by equation (93.2). In other words, the $m+1$ partial derivatives of S with respect to the $m+1$ independent variables (q, t) may be assigned arbitrary values consistent with equation (92.9). This is the characteristic property of a complete integral.

If, in addition to, or instead of, the absence of the variable t from the Hamiltonian function H, one or more of the position coördinates are absent from H, the method of procedure is just the same. Let us suppose that, in addition to the absence of t, q_1 does not appear in H. We have, then, the momentum integral $p_1 =$ constant. Denoting this constant by $p_1{}^0$, we write

$$S = -ht + p_1{}^0 q_1 + W(q_2, q_3, \cdots, q_m), \tag{93.3}$$

where now the W function does not involve either t or the coördinate q_1. Our equation (92.9) is replaced by the equation

$$h = H\left(q_2, \cdots, q_m, p_1{}^0, \frac{\partial W}{\partial q_2}, \cdots, \frac{\partial W}{\partial q_m}\right), \tag{93.4}$$

which is a partial-differential equation with one dependent variable and $m-1$ independent variables. A complete integral of this equation will involve, in addition to the two constants $(h, p_1{}^0)$, $m-2$ arbitrary constants none of which is merely additive. It is, as before, evident that if W is a complete integral of (93.4), then S, as defined by (93.3), will be a complete integral of (92.9). If the function H contained t, while q_1 remained absent, we should write, instead of (93.3), the equation

$$S = p_1{}^0 q_1 + W(q_2, \cdots, q_m, t);$$

and our differential equation (92.9) would reduce, as before, to

$$\frac{\partial W}{\partial t} = -H\left(q_2, \cdots, q_m, t, p_1{}^0, \frac{\partial W}{\partial q_2}, \cdots, \frac{\partial W}{\partial q_m}\right),$$

which is again simpler than the original equation, since it involves one less independent variable. A complete integral of the equation for W gives, as before, a complete integral S of the Hamilton-Jacobi partial-differential equation.

EXAMPLE

We shall consider first the motion of a single particle in a plane under a central force whose magnitude is a function of the distance r to the attracting center alone.

If we use polar coördinates with the origin at the attracting center, the expression for the virtual work may be written $f(r)\delta r$, so that the potential energy of the system is $V = -\int^r f(r)dr$, and there are no dissipative forces. The kinetic energy is $T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$, whence we derive $p_r = m\dot{r}$, $p_\theta = mr^2\dot{\theta}$, so that the Hamiltonian function H is

$$H = \frac{1}{2\,m}\left(p_r{}^2 + \frac{p_\theta{}^2}{r^2}\right) + V(r).$$

Since neither t nor θ appears explicitly in H, we write

$$S = -\,ht + p_\theta{}^0\theta + W(r) \tag{93.5}$$

in the Hamilton-Jacobi partial-differential equation

$$\frac{\partial S}{\partial t} + \frac{1}{2\,m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{\left(\frac{\partial S}{\partial \theta}\right)^2}{r^2}\right] + V(r) = 0.$$

We obtain in this way the ordinary differential equation for the function W of the single independent variable r,

$$\left(\frac{dW}{dr}\right)^2 + \frac{1}{r^2}\,p_\theta{}^{0\,2} = 2\,m(h - V), \tag{93.6}$$

from which W follows, by the quadrature

$$W = \int^r \frac{[2\,mr^2(h - V) - p_\theta{}^{0\,2}]^{\frac{1}{2}}}{r}\,dr. \tag{93.7}$$

The momenta along the path are given by the formulas

$$p_\theta = p_\theta{}^0, \quad p_r = \frac{\partial S}{\partial r} = \frac{dW}{dr} = \frac{[2\,mr^2(h - V) - p_\theta{}^{0\,2}]^{\frac{1}{2}}}{r};$$

and the equation of the path in the position-coördinate space, that is, the relation between the position coördinates (r, θ), is

$$\frac{\partial S}{\partial p_\theta{}^0} = \text{constant}, \quad \text{or} \quad \theta + \frac{\partial W}{\partial p_\theta{}^0} = \text{constant}.$$

The time at which the particle is at any point (r, θ) of its path is given by the final equation

$$\frac{\partial S}{\partial h} = \text{constant}, \quad \text{or} \quad t = \frac{\partial W}{\partial h} + \text{constant}.$$

In order to carry out the solution to a completion it is necessary to evaluate the integral (93.7) for W. We can tell by inspecting this integral just what kind of functions will enter the solution. For example, if the law of force toward the center is a power law $f = kr^n$, we have $V = \frac{-kr^{n+1}}{n+1}$, unless $n = -1$, when $V = -k \log r$. On making the substitution $r = \frac{1}{u}$, we see that we have to integrate an expression of the type

$$\int (a + bu^2 + cu^{-n-1})^{\frac{1}{2}} du,$$

where a, b, and c are constants. (When $n = -1$, the last term in the expression under the radical must be replaced by $c \log u$.) When the expression under the radical is of the first or second degree, the integration will not involve anything more complicated than the trigonometric or logarithmic functions. This will be the case if $n = -2$ or -3. To these cases we may add the case $n = 1$, as the substitution $u^2 = v$ will bring the quadrature to a form in which the expression under the radical is of the second degree. These are the only cases which are, in general, soluble with the aid of the elementary functions. If we look for the cases which are soluble in terms of the elliptic functions, the expression under the radical must be of the third or fourth degree at most (or must be reducible to such). This gives us the cases $n = 0, -4$, and -5, to which we may add the cases $n = 3, 5, -7$, obtainable by means of the substitution $u^2 = v$.

94. The solution of the Hamilton-Jacobi partial-differential equation in the case where the variables are separable. In the case, just treated, of the motion of a particle under a central force, the special feature of the problem which made its treatment easy was that, two of the variables being absent from the Hamiltonian function H, the *partial*-differential equation could be replaced by an *ordinary* differential equation. There is a rather wide class of problems in which a similar simplification occurs. This class is distinguished by the fact that the variables q are separable from one another in the expression on the right-hand side of (93.2). (The time variable is supposed not to enter H in the class to which we refer.) If we suppose that the kinetic energy T of the system is given by an expression of the form

$$T = \tfrac{1}{2}\Big(\sum_r \phi_r\Big)\Big[\sum_r A_r \dot{q}_r^2\Big], \tag{94.1}$$

in which each of the functions ϕ_r and A_r involves only the corresponding position coördinate q_r $(r = 1, 2, \cdots, m)$, we have the momenta p given by the formulas

$$p_s = \frac{\partial T}{\partial \dot{q}_s} = \left(\sum_r \phi_r\right) A_s \dot{q}_s;$$

and so the Hamiltonian function H is

$$H = \frac{\frac{1}{2}\left[\sum_r \frac{p_r^2}{A_r}\right]}{\sum_r \phi_r} + V.$$

In order that the variables may be separable in H, we shall have to suppose that V is expressible in the form

$$V = \frac{\sum_r \psi_r}{\sum_r \phi_r}, \tag{94.2}$$

where, like the ϕ's, each of the functions ψ_r is a function of the corresponding coördinate q_r alone. In this event the Hamilton-Jacobi partial-differential equation can, after the time is eliminated as in (93.2), be reduced to

$$\sum_r \frac{\left(\frac{\partial W}{\partial q_r}\right)^2}{A_r} = 2\sum_r (h\phi_r + \psi_r). \tag{94.3}$$

A complete integral of this equation may be found in the form

$$W = W_1 + W_2 + \cdots + W_m, \tag{94.4}$$

where each W_r is a function of the corresponding position coördinate q_r alone. In fact, we have merely to write down the ordinary differential equations

$$\left(\frac{dW_r}{dq_r}\right)^2 = 2\,A_r(h\phi_r + \psi_r) + \alpha_r, \quad (r = 1, 2, \cdots, m) \tag{94.5}$$

the α's being constants subject to the single relation

$$\sum_r \alpha_r = 0,$$

so that $m - 1$ of them may be chosen arbitrarily. Let us suppose that α_m is regarded as dependent on the other independent α's, so that it may be replaced where convenient by $-(\alpha_1+\alpha_2+\cdots+\alpha_{m-1})$. Then W is a complete integral of (94.3), depending on h and $m-1$ arbitrary constants $(\alpha_1, \cdots, \alpha_{m-1})$, each of these constants entering it through the corresponding W_r and also through W_m, and we have the relation

$$\frac{\partial W}{\partial \alpha_r}=\frac{\partial W_r}{\partial \alpha_r}+\frac{\partial W_m}{\partial \alpha_m}\frac{\partial \alpha_m}{\partial \alpha_r}=\frac{\partial W_r}{\partial \alpha_r}-\frac{\partial W_m}{\partial \alpha_m}. \quad (r=1,2,\cdots,m-1)$$

Hence the solution of the dynamic problem is implicit in the equations

$$\frac{dW_r}{dq_r}=p_r, \quad \text{or} \quad [2\,A_r(h\phi_r+\psi_r)+\alpha_r A_r]^{\frac{1}{2}}=p_r, \quad (r=1,2,\cdots,m)$$

$$\frac{\partial W_r}{\partial \alpha_r}-\frac{\partial W_m}{\partial \alpha_m}=\beta_r, \quad (r=1,2,\cdots,m-1)$$

$$t=\sum_r\frac{\partial W_r}{\partial h}+\beta_m, \tag{94.6}$$

where the β's are arbitrary constants, as are also h and the α's, the latter being, however, subject to the condition that their sum is zero. The values to be attributed to these constants are determined by the initial state of the system.

EXAMPLE

Determine the motion of a system with two position coördinates, for which $T=\frac{1}{2}(q_1^2+q_2^2)(\dot q_1^2+\dot q_2^2)$ and $V=\left(\frac{1}{q_1^2+q_2^2}\right)$.

Solution. Here $H=\frac{p_1^2+p_2^2}{2(q_1^2+q_2^2)}+\frac{1}{q_1^2+q_2^2}$, and W_1 and W_2 are given by the quadratures

$$W_1=\int^{q_1}(2\,hq_1^2+\alpha-1)^{\frac{1}{2}}dq_1, \quad W_2=\int^{q_2}(2\,hq_2^2-\alpha-1)^{\frac{1}{2}}dq_2,$$

where α is an arbitrary constant. The paths of the system in the (q_1, q_2) space are given by $\frac{\partial W_1}{\partial \alpha}+\frac{\partial W_2}{\partial \alpha}=\beta$. This gives

$$\int^{q_1}\frac{dq_1}{(q_1^2-l^2)^{\frac{1}{2}}}+\int^{q_2}\frac{dq_2}{(q_2^2-m^2)^{\frac{1}{2}}}=\text{constant},$$

where we have taken $l^2=\frac{1-\alpha}{2\,h}$, $m^2=\frac{1+\alpha}{2\,h}$. This gives

$$\cosh^{-1}\frac{q_1}{l}+\cosh^{-1}\frac{q_2}{m}=\text{constant}.$$

On taking the hyperbolic sine of this equation, we get

$$\frac{q_1q_2}{lm}+\frac{[(q_1^2-l^2)(q_2^2-m^2)]^{\frac{1}{2}}}{lm}=\text{constant}.$$

Upon clearing of radicals by squaring, this can be put in the form

$$m^2 q_1^2 + 2\, lm \cos\gamma\; q_1 q_2 + l^2 q_2^2 = l^2 m^2 \sin^2\gamma,$$

where γ is an arbitrary constant. On a Cartesian-plane diagram, with q_1 and q_2 as rectangular coördinates, these curves are concentric ellipses.

95. The problem of two gravitating centers. Here we wish to determine the motion of a particle in a plane under the influence of two centers of attraction situated at two fixed points in the plane and attracting according to the inverse-square law. This problem is of interest as an introduction to the three-body problem, which is so important in celestial mechanics. It affords a good illustration of the method of separation of variables. Taking the x-axis along the join of the attracting centers, and the y-axis halfway between them, the coördinates of the two attracting centers will be $(\pm c, 0)$, $2c$ being the distance between them. Let us consider the system of confocal conics

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} = 1, \quad a^2 - b^2 = c^2,$$

with foci at the two attracting centers. Through any point (x, y) of the plane there pass two conics of this confocal system, the corresponding values of θ being λ and μ. These are the roots of the quadratic equation in θ

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} = 1,$$

so that we have the identity in θ

$$1 - \frac{x^2}{a^2+\theta} - \frac{y^2}{b^2+\theta} = \frac{(\theta-\lambda)(\theta-\mu)}{(a^2+\theta)(b^2+\theta)},$$

where (λ, μ) are regarded as functions of (x, y) in this identity. Upon multiplying through by $a^2+\theta$ and then setting $\theta = -a^2$, we find

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)}{a^2-b^2},$$

and, similarly, on multiplying through by $b^2+\theta$ and setting $\theta = -b^2$, we find

$$y^2 = \frac{-(b^2+\lambda)(b^2+\mu)}{a^2-b^2}.$$

It will be convenient to use, as the position coördinates (q_1, q_2) of the particle, the semiaxes — major and transverse respectively — of the ellipse and hyperbola of the confocal system passing through (x, y). (If we suppose $\lambda > \mu$, $b^2 + \mu$ is negative, from the expression for y^2; so that the root $\theta = \mu$ gives a hyperbola, the other root λ giving the ellipse.) We have, then, $q_1{}^2 = a^2 + \lambda$, $q_2{}^2 = a^2 + \mu$; so that $x^2 = \frac{q_1{}^2 q_2{}^2}{c^2}$. We may write $x = \frac{q_1 q_2}{c}$ if we agree to regard q_2 as negative when x is negative. For y^2 we find at once the expression $\frac{(q_1{}^2 - c^2)(c^2 - q_2{}^2)}{c^2}$, and we readily derive the distance r_1 of the point (x, y) from the attracting center $(c, 0)$; thus $r_1{}^2 = x^2 - 2cx + c^2 + y^2 = (q_1 - q_2)^2$, whence $r_1 = q_1 - q_2$. This value of the radical holds for all points (x, y) on account of the convention as to the sign of q_2. Similarly we find, for the distance r_2 to the other attracting center, the result $r_2 = q_1 + q_2$. The quantities q_1 and q_2 are known as the elliptic coördinates of the point (x, y). These expressions for r_1 and r_2 could have been written down at once if use had been made of the elementary geometric properties of the ellipse and hyperbola (we refer to the constancy of the sum or difference of the focal radii).

Now the kinetic energy of the moving particle is $T = \frac{1}{2} mv^2$, m being its mass and $v^2 = \dot{x}^2 + \dot{y}^2$ the square of its velocity. On substituting for x and y their values in terms of q_1 and q_2 given above, we find

$$T = \frac{1}{2} m(q_1{}^2 - q_2{}^2)\left[\frac{\dot{q}_1{}^2}{q_1{}^2 - c^2} + \frac{\dot{q}_2{}^2}{c^2 - q_2{}^2}\right].$$

The potential energy is given by

$$V = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} = -\frac{\mu_1}{q_1 - q_2} - \frac{\mu_2}{q_1 + q_2},$$

where μ_1 and μ_2 are constants. It appears at once that the variables are separable, so that the determination of a complete integral of the Hamilton-Jacobi partial-differential equation may be reduced to quadratures. We write $S = -ht + W_1 + W_2$, where h is the energy constant and W_1 and W_2 are determined by the equations

$$(q_1{}^2 - c^2)\left(\frac{dW_1}{dq_1}\right)^2 = 2\,mq_1(\mu_1 + \mu_2 + hq_1) + \alpha,$$

$$(c^2 - q_2{}^2)\left(\frac{dW_2}{dq_2}\right)^2 = 2\,mq_2(\mu_1 - \mu_2 - hq_2) - \alpha,$$

where α is an arbitrary constant. The actual path of the particle in its motion in the plane is given in terms of the elliptic coördinates by the equation

$$\frac{\partial W_1}{\partial \alpha} + \frac{\partial W_2}{\partial \alpha} = \beta,$$

where β is an arbitrary constant. The time at which any point on the path is occupied is found by putting t equal to the sum of the derivatives of W_1 and W_2 with respect to the energy constant h, plus an additive constant. It will be observed that the quadratures involved in the determination of W_1 and W_2 bring in the radical of a polynomial of the third degree. The solution therefore requires the use of elliptic integrals.

It will be observed that if, in addition to the two centers attracting according to the inverse-square law, there were an attracting center located at the mid-point of their join, attracting according to the direct-distance law, there would be added to V a term of the type $-\frac{1}{2}\mu_3(x^2 + y^2)$; and since, in terms of the elliptic coördinates, this is $\frac{1}{2}\mu_3(c^2 - q_1{}^2 - q_2{}^2)$, the variables are again seen to be separable. Similarly, forces parallel to either of the coördinate axes, and each inversely proportional to the cube of the distance from the other axis, may be added without affecting the separability of the variables. The problem of the motion of a particle, under the attraction of two fixed centers, is treated in detail in the treatises on celestial mechanics; and we may refer in particular to Charlier's "Mechanik des Himmels," Vol. I, chap. iii.

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EXERCISES

1. Discuss the motion of a particle on a surface of revolution when the potential energy is a function of the distance from the axis alone. Use the Hamilton-Jacobi method.

2. The velocity of a particle in its orbit is found to vary as the inverse square of its distance from a fixed point. Apply the principle of least action to find the orbit, and thence the law of attraction. Deduce the same results from the law of conservation of energy.

3. Discuss the motion of a projectile by the Hamilton-Jacobi method.

4. Show that the transformation defined by the equations

$$Q_1 = q_1^2 + \lambda^2 p_1^2, \qquad Q_2 = q_2^2 + \lambda^2 p_2^2,$$

$$P_1 = \frac{\arctan\left(\dfrac{q_1}{\lambda p_1}\right)}{-\arctan\left(\dfrac{q_2}{\lambda p_2}\right)}, \quad P_2 = \lambda \arctan\left(\frac{q_2}{\lambda p_2}\right)$$

is a contact transformation, and that it reduces the dynamic system whose Hamiltonian function is $\frac{1}{2}(p_1^2 + p_2^2 + \lambda^{-2}q_1^2 + \lambda^{-2}q_2^2)$ to the dynamic system whose Hamiltonian function is Q_2.

CHAPTER XIV

NONHOLONOMIC SYSTEMS; THE PRINCIPLE OF LEAST CONSTRAINT

96. Nonholonomic dynamic systems. In the discussion of generalized coördinates we have said that the coördinates $(q_1, q_2, \cdots, q_k)$ of a dynamic system are distinct when there exists no functional relation $f(q_1, q_2, \cdots, q_k; t) = 0$, possibly involving the time, connecting them. It was remarked that it is frequently convenient, for reasons of symmetry or ease of manipulation, to introduce a number of superfluous coördinates; and the method of undetermined multipliers for dealing with these was explained. When the coördinates are not distinct, the geometric, or virtual, differentials δq are connected by a number of relations

$$\frac{\partial f}{\partial q_1}\,\delta q_1 + \cdots + \frac{\partial f}{\partial q_k}\,\delta q_k = 0,$$

obtained by differentiating the relations $f(q_1, \cdots, q_k; t) = 0$ connecting the generalized coördinates, the time being regarded as constant in the differentiation. The coördinate velocities are connected with each other by the corresponding linear relations

$$\frac{\partial f}{\partial q_1}\,\dot{q}_1 + \cdots + \frac{\partial f}{\partial q_k}\,\dot{q}_k + \frac{\partial f}{\partial t} = 0.$$

The important thing to notice is that these relations are integrable; that is, the finite relations $f(q_1, q_2, \cdots, q_k, t) = 0$ can be deduced from them. Now it sometimes happens that the coördinate velocities $\dot{q}$ are connected by one or more linear relations of the type

$$a_1\dot{q}_1 + a_2\dot{q}_2 + \cdots + a_k\dot{q}_k + b_k = 0, \tag{96.1}$$

which are nonintegrable. In other words, it is impossible to set up a number of finite relations $f(q_1, q_2, \cdots, q_k; t) = 0$, connecting the coördinates and the time, which are equivalent to the relations (96.1) connecting the coördinate velocities and the time. The coördinates are, accordingly, still distinct; that is, it is impossible to find a fewer number of coördinates which will serve to describe the dynamic system. If there are p distinct linear relations (96.1) connecting the velocities of the distinct coördinates, the system will be said to have $k - p$ degrees of freedom and will be called nonholonomic, a holonomic system being one where there are no nonintegrable relations connecting the coördinate velocities. In a holonomic system, therefore, the number of degrees of freedom is the same as the number of distinct coördinates, whereas in a nonholonomic system the number of degrees of freedom is less than the number of distinct coördinates by the number of independent nonintegrable relations connecting the coördinate velocities.

The simplest examples of nonholonomic systems occur in problems dealing with the rolling motion of one body upon another. If there is no slipping, we have to express the fact that the relative velocity at the point of contact is zero. Consider, for example, the simple case of a sphere rolling upon a plane. Taking the plane as the z-plane of a Cartesian reference frame, the sphere has five distinct coördinates: the two Cartesian coördinates (x, y) of the projection of its center on the plane on which it is rolling, and the three Eulerian angles (θ, ϕ, ψ) of § 23. To express the fact that the drag velocity of the point of contact of the sphere with the plane is zero, we have the two relations

$$\dot{x} - a\omega_\eta = 0, \quad \dot{y} + a\omega_\xi = 0.$$

Here a is the radius of the sphere, and $(\omega_\xi, \omega_\eta)$ are given in (26.11). This yields the two nonintegrable relations

$$\begin{aligned} \dot{x} - a\dot{\theta}\cos\phi - a\sin\theta\sin\phi\,\dot{\psi} &= 0, \\ \dot{y} - a\dot{\theta}\sin\phi + a\sin\theta\cos\phi\,\dot{\psi} &= 0 \end{aligned}$$

connecting the five coördinate velocities $(\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}, \dot{\psi})$. The problem is nonholonomic, with three degrees of freedom.

97. Appell's equations for a dynamic system. We shall now outline a mode of treatment of the motion of a dynamic system, due to Appell, which has the advantage that it is equally applicable to holonomic and nonholonomic systems. Nonholonomic systems may be treated by Lagrange's method if suitable modifications are made; but Appell's method applies without any modification to either case. The new method has also the advantage, from a physical point of view, that it deals directly with the accelerations of the various points of the system, and not immediately with the velocities which enter so prominently, through the kinetic energy, in Lagrange's discussion. The fundamental laws of dynamics tell us that the accelerations are of primary importance, the velocities being secondary. On the other hand, it must be admitted that, as an offset to these advantages, the mathematical difficulties are usually greater in Appell's method than in Lagrange's method, — at any rate in the simpler cases of holonomic systems.

In order to treat any kind of situation that may arise, we shall suppose that, in addition to the k distinct coördinates $(q_1, q_2, \cdots, q_k)$ of the system, certain superfluous coördinates $q_{k+1}, \cdots, q_{k+s}$ may appear in the discussion. Let us denote, for the moment, by $(\pi_1, \pi_2, \cdots, \pi_s)$ certain linear expressions in the differentials of the coördinates $(q_1, \cdots, q_{k+1}, \cdots, q_{k+s})$ and, possibly, also the time differential dt. Then we may have p distinct relations of the type

$$\begin{aligned} a_{r1}\,dq_1 + \cdots + a_{rk}\,dq_k + a_{r,k+1}\pi_1 + a_{r,k+2}\pi_2 \\ + \cdots + a_{r,k+s}\pi_s + a_r\,dt = 0, \quad (r = 1, 2, \cdots, p) \quad (97.1) \end{aligned}$$

connecting the $k+s+1$ expressions $(dq_1, \cdots, dq_k;\ \pi_1, \cdots, \pi_s;\ dt)$. The system has $k+s$ coördinates, of which s are superfluous, and $k+s-p$ degrees of freedom. If $p = s$, all the relations (97.1) are integrable, and we have merely the holonomic case with superfluous coördinates. If $p = s = 0$, we have the simplest case, holonomic with distinct coördinates. In the nonholonomic case the number p of relations (97.1) is greater than s, and the number $n = k + s - p$ of degrees of freedom is less than the number k of distinct coördinates.

Corresponding to the relations (97.1) we shall have a set of p relations on the virtual, or geometric, differentials δq obtained by setting $dt = 0$ wherever it occurs in equations (97.1). The p relations obtained in this way being independent of one another, it is possible, from (97.1), to solve for p of the quantities $(dq_1, dq_2, \cdots, dq_k;\ \pi_1, \pi_2, \cdots, \pi_s)$ in terms of the remaining $k+s-p = n$. Denoting these n quantities, in terms of which the remaining p's are supposed to be linearly expressed, by $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, and the others by $(\lambda_{n+1}, \cdots, \lambda_{n+p})$, we have the p equations

$$\lambda_{n+r} = \alpha_{r1}\lambda_1 + \cdots + \alpha_{rn}\lambda_n + \alpha_r\, dt. \quad (r = 1, 2, \cdots, p) \tag{97.2}$$

In these expressions the values to be assigned to the n quantities $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ are quite arbitrary. Now the Cartesian coördinates (x, y, z) of any particle of the system being given in terms of the generalized coördinates q by equations

$$x = x(q_1, \cdots, q_{k+s}, t) \text{ etc.,}$$

we have
$$dx = \frac{\partial x}{\partial q_1}\, dq_1 + \cdots + \frac{\partial x}{\partial t}\, dt, \text{ etc.};$$

and by means of (97.2) we are able to write dx in the form

$$dx = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n + a\, dt.$$

Similarly,
$$dy = b_1\lambda_1 + b_2\lambda_2 + \cdots + b_n\lambda_n + b\, dt \tag{97.3}$$

and
$$dz = c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n + c\, dt.$$

The geometric differentials being obtained from these kinematic differentials by dropping the terms involving dt, we have, for the virtual work of the forces acting on the system,

$$\Sigma(X\,\delta x + Y\,\delta y + Z\,\delta z) = \Gamma_1\lambda_1 + \Gamma_2\lambda_2 + \cdots + \Gamma_n\lambda_n, \tag{97.4}$$

where
$$\Gamma_r = \Sigma(Xa_r + Yb_r + Zc_r). \quad (r = 1, 2, \cdots, n) \tag{97.5}$$

The principle of virtual work gives

$$\Sigma m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z) = \Sigma(X\,\delta x + Y\,\delta y + Z\,\delta z), \tag{97.6}$$

where, if the system is assumed frictionless, the right-hand side is the virtual work of the active forces acting on the system. Since the n quantities $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ are arbitrary, we can equate the coefficients of each of these quantities on both sides of (97.6). Let us find the coefficient of λ_r on the left-hand side of (97.6). From (97.3) we have for this coefficient the expression

$$\Sigma m(\ddot{x}a_r + \ddot{y}b_r + \ddot{z}c_r), \tag{97.7}$$

but this can be put in a much better form. Thus, if we denote by μ_1 the quotient of λ_1 by dt, with a similar notation for the other λ's, we have

$$\dot{x} = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n + a;$$

whence

$$\ddot{x} = a_1\dot{\mu}_1 + a_2\dot{\mu}_2 + \cdots + a_n\dot{\mu}_n + \dot{a} + \dot{a}_1\mu_1 + \cdots + \dot{a}_n\mu_n.$$

Hence, if we regard $\ddot{x}$ as a function of the variables $(\mu, \dot{\mu})$, $a_r = \frac{\partial \ddot{x}}{\partial \dot{\mu}_r}$, and (97.7) takes the form

$$\frac{\partial S}{\partial \dot{\mu}_r},$$

where

$$2\,S = \Sigma m(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2). \tag{97.8}$$

The expression S is formed from the acceleration vectors of the system in exactly the same way as the kinetic energy is formed from the velocity vectors of the system. For this reason it is referred to as the "energy of the accelerations." The differential equations governing the motion of the system now have the form

$$\frac{\partial S}{\partial \dot{\mu}_r} = \Gamma_r. \quad (r = 1, 2, \cdots, n) \tag{97.9}$$

The expressions $\dot{\mu}_r$ are linear functions of the coördinate accelerations. Any terms in S depending only on the coördinate velocities may be neglected when we are writing out explicitly equations (97.9), since the μ, or, equivalently, the coördinate velocities, are regarded as constant in the differentiation with respect to $\dot{\mu}$.

98. The principle of least constraint. Equations (97.9) lead at once to a method of statement of the laws governing the motion of a dynamic system which is known as the Gaussian principle of least constraint, or, alternatively, as the Hertzian principle of least curvature. The position and velocity of each particle of a dynamic system at a certain instant being known, it is desired to know the acceleration that each particle has at that instant, the forces acting on the system being given. It is at once apparent, from equations (97.9), that the accelerations of the various particles of the system are such that the partial derivatives of the function

$$S - \Gamma_1\dot{\mu}_1 - \Gamma_2\dot{\mu}_2 - \cdots - \Gamma_n\dot{\mu}_n \tag{98.1}$$

with respect to each $\dot{\mu}$ are zero, the coördinate velocities being regarded as constant in the differentiation. Now

$$\Sigma(X\ddot{x} + Y\ddot{y} + Z\ddot{z})$$

is, by (97.3), equal to

$$\Gamma_1\dot{\mu}_1 + \Gamma_2\dot{\mu}_2 + \cdots + \Gamma_n\dot{\mu}_n + \Sigma(Xa + Yb + Zc).$$

On taking the time derivative of each of these modes of expressing the rate at which the forces acting are doing work on the system, we see that the difference between

$$\Sigma(X\ddot{x} + Y\ddot{y} + Z\ddot{z})$$

and

$$\Gamma_1\dot{\mu}_1 + \Gamma_2\dot{\mu}_2 + \cdots + \Gamma_n\dot{\mu}_n$$

depends merely on the position and velocity of the various particles of the system, and not at all on their accelerations. We see, therefore, that the accelerations assumed by the various particles of the system are such as to give the expression in (98.1), or, equivalently, the expression

$$S - \Sigma(X\ddot{x} + Y\ddot{y} + Z\ddot{z}), \tag{98.2}$$

a stationary value when regarded as a function of the accelerations alone, the position, time, and velocity coördinates not being allowed to vary. Instead of the function in (98.2) we may take the slightly different function, R, defined by

$$2R = \Sigma m\left[\left(\ddot{x} - \frac{X}{m}\right)^2 + \left(\ddot{y} - \frac{Y}{m}\right)^2 + \left(\ddot{z} - \frac{Z}{m}\right)^2\right], \tag{98.3}$$

as R differs from the expression in (98.2) merely by

$$\frac{1}{2}\sum \frac{1}{m}(X^2 + Y^2 + Z^2),$$

which does not involve the accelerations. Each term in the summation (98.3) is the quotient, by the mass m of the particle in question, of the square of the magnitude of the vector difference between the active force on that particle and its kinetic reaction, that is, the product of its mass by its acceleration. If all the particles were free, each of these vector differences would be zero, and R would have the value zero. When the particles are not all free, R will have a positive value; and we may regard R as measuring in a certain sense the "constraint" to which the system is subjected by its connections. The accelerations assumed by the various particles of the system under the given forces, are, then, such as to give the constraint a stationary value when regarded as a function of the accelerations alone. This is known as the principle of least constraint.

As an example of this method of treating dynamic problems, let us consider the motion of a particle in a plane. Using plane polar coördinates (r, θ), the problem is holonomic with two degrees of freedom, $k = 2$, $s = 0$, $p = 0$, $n = k$. Let us adopt, as the independent quantities (λ_1, λ_2) in terms of which dr and $d\theta$ may be expressed, $\lambda_1 = dr$, $\lambda_2 = r^2 d\theta$. Then $\mu_1 = \dot{r}$, $\mu_2 = r^2\dot{\theta}$, and we have, from $x = r\cos\theta$,

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = \mu_1\cos\theta - \frac{\mu_2\sin\theta}{r},$$

$$\ddot{x} = \dot{\mu}_1\cos\theta - \frac{\dot{\mu}_2\sin\theta}{r} + \text{terms independent of the accelerations.}$$

Similarly, $\ddot{y} = \dot{\mu}_1\sin\theta + \dfrac{\dot{\mu}_2\cos\theta}{r}$ + unessential terms.

Hence $S = \dfrac{m}{2}\left(\dot{\mu}_1{}^2 + \dfrac{\dot{\mu}_2{}^2}{r^2}\right)$ + unessential terms.

The virtual work being written in the form

$$X\,\delta x + Y\,\delta y = R\,\delta r + \Theta\, r\delta\theta,$$

we have

$$\Gamma_1 = R, \quad \Gamma_2 = \frac{\Theta}{r},$$

and the equations of motion are

$$m\dot{\mu}_1 = R, \quad m\dot{\mu}_2 = r\Theta.$$

If μ_2 is constant, — that is, if the rate of description of areas is constant, — the second of these two equations shows that $\Theta = 0$, so that the active force is a central one.

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EXERCISES

1. Use Appell's equations to discuss the motion of a circular hoop rolling on rough ground but not necessarily in a vertical plane.

2. Investigate the form of the energy of the accelerations of a rigid body turning about a fixed point. Deduce Euler's equations.

3. A uniform sphere rolls on a perfectly rough horizontal plane under forces whose resultant passes through the center. Show that the motion of its center is the same as that of a particle acted on by the same forces reduced in the ratio 5 : 7.

4. Determine the motion of a perfectly rough sphere of radius a and mass m which rolls on a fixed sphere of radius b, the only external force being gravity.

5. A circular disk, capable of motion about a vertical axis through its center perpendicular to its plane, is set in motion with angular velocity ω. A rough uniform sphere is gently placed on any point of the disk other than the center. Prove that the sphere will describe a circle on the disk, and that the disk will revolve with angular velocity $\frac{7I\omega}{7I+2mr^2}$, where I is the moment of inertia of the disk about its axis, m is the mass of the sphere, and r is the radius of the circle traced out.

6. A perfectly rough sphere of radius a is made to rotate about a vertical diameter, which is fixed, with a constant angular velocity n. A uniform sphere of radius b is placed on it at a point distant $a\alpha$ from the highest point: investigate the motion and determine in any position the angular velocity of the sphere. Show that the sphere will leave the rotating sphere when the point of contact is at an angular distance θ from the vertex, where $\cos\theta = \frac{10}{17}\cos\alpha + \frac{4a^2n^2\sin^2\alpha}{119(a+b)g}$.

7. If T, V, and S denote, respectively, the kinetic and potential energies and the energy of the accelerations of a dynamic system, show that $\frac{d^2V}{dt^2}+S$ differs from

$$\sum\frac{1}{2m}\left[\left(m\ddot{x}+\frac{\partial V}{\partial x}\right)^2+\left(m\ddot{y}+\frac{\partial V}{\partial y}\right)^2+\left(m\ddot{z}+\frac{\partial V}{\partial z}\right)^2\right]$$

by a quantity which does not involve the accelerations; and hence that $\frac{d^2T}{dt^2}-S$ is stationary when the accelerations have the values corresponding to the actual motion, as compared with all motions which are consistent with the constraints and satisfy the same integral of energy, and which have the same values of the coördinates and velocities at the instant considered.

CHAPTER XV

GENERAL METHODS OF INTEGRATION; POISSON BRACKETS

99. First integrals of the dynamic equations. We have seen, in (80.4), that the laws governing the motion of any holonomic dynamic system may be expressed by means of $2m$ differential equations of the first order, where m is the number of degrees of freedom of the system. In the general case these equations are

$$\frac{dq_s}{dt} = \frac{\partial K}{\partial p_s}, \quad \frac{dp_s}{dt} = -\frac{\partial K}{\partial q_s} + Q_s, \quad (s = 1, 2, \cdots, m)$$

where the q's and the p's are, respectively, the coördinates and momenta of the system, the Q's are the generalized-force components, and K is defined by (80.2). In the case where the forces acting on the system are derivable from a potential function V the equations take the simpler form

$$\frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}, \quad \frac{dp_s}{dt} = -\frac{\partial H}{\partial q_s}, \quad (s = 1, 2, \cdots, m)$$

where $H = K + V$. (Evidently it is definitely understood here that the function V does not involve the coördinate velocities. It is a function of the position coördinates q and, possibly, of the time. If the coördinate velocities $\dot{q}$ entered it, it would involve the momenta p, and $\frac{\partial K}{\partial p_s}$ would not be the same as $\frac{\partial H}{\partial p_s}$.) In the case of a natural dynamic system, $K = T$, and H is the sum of the kinetic and potential energies of the system. In any event, the solution of the dynamic problem is obtained by integrating $2m$ first-order differential equations of the type

$$\frac{dq_1}{\bar{Q}_1} = \frac{dq_2}{\bar{Q}_2} = \cdots = \frac{dq_m}{\bar{Q}_m} = \frac{dp_1}{P_1} = \cdots = \frac{dp_m}{P_m} = dt.$$

This series of equalities is unaffected if we multiply each expression in the line by the same function $\left(\frac{1}{T}, \text{say}\right)$ of (q, p, t), so that our equations can always be put in the form

$$\frac{dq_1}{\bar{Q}_1} = \cdots = \frac{dq_m}{\bar{Q}_m} = \frac{dp_1}{P_1} = \cdots = \frac{dp_m}{P_m} = \frac{dt}{T} = d\theta. \qquad (99.1)$$

(The $\bar{Q}$'s and P's in this series of equalities are the products of the $\bar{Q}$'s and P's in the preceding series by the arbitrary function T, and θ is an arbitrary parameter, or independent variable. This parameter may be varied at will by varying the choice of the arbitrary function T.) The problem is to express the state coördinates (q, p, t) of the system in terms of the parameter θ and the initial state coördinates (q_0, p_0, t_0), that is, the values assumed by (q, p, t) when θ is assigned some convenient numeric value θ_0. In the case of a natural dynamic system, which is the case in which we are mainly interested, we note the values

$$\bar{Q}_s = \frac{\partial H}{\partial p_s}, \quad P_s = -\frac{\partial H}{\partial q_s}, \quad T = 1. \quad (s = 1, 2, \cdots, m) \qquad (99.2)$$

For convenience of reference let us rewrite equations (99.1) in the form

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} = d\theta, \qquad (99.3)$$

so that $2m + 1 = n$ and

$$\begin{aligned} q_s &= x_s, & p_s &= x_{m+s}, & t &= x_n, \\ \bar{Q}_s &= X_s, & P_s &= X_{m+s}, & T &= X_n. \end{aligned} \quad (s = 1, 2, \cdots, m) \qquad (99.4)$$

If we arbitrarily assign the values $(y_1, y_2, \cdots, y_n)$ to the dependent variables $(x_1, x_2, \cdots, x_n)$ when $\theta = 0$, equations (99.3) define the x's uniquely in terms of θ and the y's, it being understood that the n X's are functions of the n x's. In other words, the solution of equations (99.3) is of the form

$$x_s = f_s(\theta; y_1, y_2, \cdots, y_n). \quad (s = 1, 2, \cdots, n) \qquad (99.5)$$

For a given set of numeric values y, these equations may be said to define a curve in a space of n dimensions in which the coördinates are the x's. For each numeric value assigned to θ, equations (99.5) define a point on this curve; and, in particular, when $\theta = 0$ we have the point y. We call these curves integral curves of equations (99.3), so that there is one and only one integral curve through each point y of the space of n dimensions in which the coördinates are the x's.

As we move along any one of these integral curves the x's themselves and, in consequence, any function of them will, in general, take variable numeric values. There are, however, certain special functions of the x's which do not vary as we move along any integral curve, and these functions are known as first integrals of the differential equations (99.3). The criterion for first integrals is easy to set down. In fact, if $F(x_1, x_2, \cdots, x_n)$ is a first integral, it must have a constant value when we substitute for each of the x's its expression in terms of the parameter θ and the y's given in (99.5). In other words, the resulting function of θ and the y's must not actually involve θ. The test for this is that the derivative of F with respect to θ, the y's being held constant, must vanish identically. Differentiating F with respect to θ, and using the information $\frac{dx_s}{d\theta} = X_s$, we have the result that $F(x_1, x_2, \cdots, x_n)$ is a first integral of equations (99.3) if, and only if, it satisfies the homogeneous first-order partial-differential equation

$$X_1 \frac{\partial F}{\partial x_1} + X_2 \frac{\partial F}{\partial x_2} + \cdots + X_n \frac{\partial F}{\partial x_n} = 0. \qquad (99.6)$$

We shall have to refer to this equation so frequently that it will be convenient to use the abbreviation $X(F)$ for the left-hand member of it.

It is at once apparent that if F is a first integral, any function $\phi(F)$, say, is also a first integral. For if the argument F remains constant along any integral curve, then the function $\phi(F)$ will remain constant along any integral curve. It is also evident that there cannot be more than $n - 1$ functionally independent first integrals. For if there were n first integrals $(F_1, F_2, \cdots, F_n)$, we should have n homogeneous linear equations of the type (99.6); and for these to be consistent (the X's not all vanishing) the determinant of the coefficients would have to vanish identically. But this determinant is nothing but the Jacobian of the n functions $(F_1, F_2, \cdots, F_n)$ with respect to the x's, and the vanishing of this Jacobian implies a functional relationship $\phi(F_1, F_2, \cdots, F_n) = 0$ connecting the n first integrals $(F_1, F_2, \cdots, F_n)$.

That there actually exist $n - 1$ functionally independent first integrals and no fewer is readily seen as follows. Let us suppose equations (99.5) solved for the y's in terms of the parameter θ and the x's, thus:

$$y_s = g_s(\theta;\ x_1, \cdots, x_n). \quad (s = 1, 2, \cdots, n) \tag{99.7}$$

Solving for θ from one of these equations, — the first, let us say, — and substituting the resulting value in the remaining equations, we have $n - 1$ equations of the type

$$y_s = h_s(y_1;\ x_1, \cdots, x_n). \quad (s = 2, 3, \cdots, n) \tag{99.8}$$

The functions h_s are first integrals, since h_2, for example, has the constant value y_2 along the integral curve determined by the initial point $(y_1, y_2, \cdots, y_n)$. That these $n - 1$ first integrals $(h_2, \cdots, h_n)$ are functionally independent is evident from the fact that the values assigned to $(y_2, y_3, \cdots, y_n)$ may be chosen arbitrarily. Conversely, let us suppose that we know $n - 1$ first integrals of the differential equations (99.3), and let us denote these first integrals by $(F_1, F_2, \cdots, F_{n-1})$. Upon substituting arbitrarily chosen numeric values y for the x's, these first integrals take certain numeric values which we may denote by $(C_1, C_2, \cdots, C_{n-1})$. Then the $n - 1$ distinct relations

$$F_s(x_1, x_2, \cdots, x_n) = C_s \quad (s = 1, 2, \cdots, n - 1)$$

among the x's imply that only one of the x's can be chosen arbitrarily and that the others are functions of this one. Expressed in a more symmetrical manner, each of the x's is a function of a single independent parameter θ. Implicit in these functions of the parameter θ are, of course, the n arbitrary constants y.

That the functions of θ obtained in this way actually satisfy the differential equations (99.3) is at once evident. For on differentiating the equations

$$F_s(x_1, x_2, \cdots, x_n) = C_s$$

with respect to θ, through the x's, we have

$$\frac{\partial F_s}{\partial x_1}\frac{dx_1}{d\theta} + \cdots + \frac{\partial F_s}{\partial x_n}\frac{dx_n}{d\theta} = 0. \quad (s = 1, 2, \cdots, n - 1)$$

But we are given at the start the $n - 1$ relations $X(F_s) = 0$; and since the $n - 1$ functions F are functionally independent (so that at least one of the determinants of order $n - 1$ out of the Jacobian matrix of the $n - 1$ F's with respect to the n x's does not vanish), these two sets of $n-1$ equations must yield the same values for the ratios of the $\frac{dx_r}{d\theta}$, in the first case. and of the X_r, in the second case. We have, then,

$$\frac{dx_1}{d\theta}:\frac{dx_2}{d\theta}:\cdots:\frac{dx_n}{d\theta}=X_1:X_2:\cdots:X_n.$$

To secure $\frac{dx_1}{d\theta}=X_1$, etc., it is merely required to make, if necessary, a change of parameter in the equations (99.3). The effect of this change of parameter is to multiply each of the X's by the same function of the x's.

We see, then, that the solution of the equations (99.3) implies and is implied by a knowledge of $n-1$ functionally independent first integrals. Expressed geometrically, the equations (99.3) assign a direction to each point x of our representative space. The problem before us is to collect these directions into curves, so that at each point the direction is tangent to the integral curve through it. A first integral F, when set equal to a constant C, is a spread of $n-1$ dimensions made up of integral curves; so that if an integral curve has a single point in common with the spread, the entire integral curve lies in the spread. If we know the integral curves, we can find the families of integral spreads; and, conversely, if we have $n-1$ functionally distinct families of integral spreads, these give, by their mutual intersections, the integral curves. For example, if $n=3$ and $X_1=0$, $X_2=0$, $X_3=1$, the integral curves are straight lines parallel to the z-axis. The equation determining the first integrals is simply $X(F)$ or $\frac{\partial F}{\partial z}=0$, of which $F_1=x$ and $F_2=y$ are two functionally distinct first integrals. The integral curves may be found as the intersections of the two families of integral spreads $x=C_1$, $y=C_2$.

Assuming that the problem has been solved, — that is, that we are in possession of $n-1$ functionally distinct first integrals, — we may introduce a change of variables which will give the equations (99.3) a particularly simple standard form. All we have to do is to set

$$F_1=z_1,\quad F_2=z_2,\quad\cdots,\quad F_{n-1}=z_{n-1}. \tag{99.9}$$

We can solve these equations for $n-1$ of the x's in terms of the remaining x and the z's. In this way we obtain a new set of differential equations in which the dependent variables are the $n-1$ z's and one x. Let us suppose that x_n is the remaining x, the others having been eliminated as indicated. Since $(F_1, F_2, \cdots, F_{n-1})$ are first integrals, the $n-1$ z's are constant along an integral curve; and so our differential equations are

$$\frac{dz_1}{d\theta} = 0, \quad \cdot\cdot, \quad \frac{dz_{n-1}}{d\theta} = 0, \quad \frac{dx_n}{d\theta} = \overline{X}_n,$$

where the bar has been placed over X_n to indicate that it is expressed in terms of the $n-1$ constants z and the one variable x_n. Introducing the new variable z_n defined by

$$z_n = \int \frac{dx_n}{\overline{X}_n},$$

our differential equations take the form

$$\frac{dz_1}{d\theta} = 0, \quad \cdots, \quad \frac{dz_{n-1}}{d\theta} = 0, \quad \frac{dz_n}{d\theta} = 1,$$

which may be written in the form

$$\frac{dz_1}{0} = \frac{dz_2}{0} = \cdots = \frac{dz_{n-1}}{0} = \frac{dz_n}{1} = d\theta. \tag{99.10}$$

Expressed geometrically, the integral curves are straight lines parallel to the z_n axis, and the integral spreads are "hyperplanes" parallel to the coördinate hyperplanes $z_1 = 0, \cdots, z_{n-1} = 0$, assuming, of course, that the z's are rectangular Cartesian coördinates in the representative space.

Even if we are not in possession of the requisite number $n-1$ of independent first integrals, we can proceed as above and reduce the order of difficulty of the problem without, perhaps, being able to solve it completely. For example, let us suppose that we know two distinct first integrals F_1 and F_2. Setting $F_1 = z_1$, $F_2 = z_2$, we can eliminate two of the x's (x_1 and x_2, say) from our problem; and there remains the integration of the $n-2$ differential equations

$$\frac{dx_3}{\overline{X}_3} = \cdots = \frac{dx_n}{\overline{X}_n} = d\theta,$$

where the bars have been placed above X_3 etc. to indicate that they are now expressed in terms of the two constants z_1 and z_2 and the $n-2$ variables $(x_3, x_4, \cdots, x_n)$. The order of difficulty of the problem is considerably less than it was previously, since we have now only $n-2$ equations instead of n.

From what has been said it is clear that if a problem in dynamics proves too difficult for an immediate solution, a systematic mode of attack is to search for first integrals. If the number of degrees of freedom of the system is m, the number of equations is $2m+1$; and a knowledge of $2m$ first integrals is necessary for the complete solution of the problem. If, however, the system is a natural one, — that is, one such that the time variable t does not enter H, so that it is absent from the denominators $\frac{\partial H}{\partial p_s}$, $-\frac{\partial H}{\partial q_s}$ of the canonical equations, — we may omit until the end the equation involving t, and need consider only the $2m$ equations

$$\frac{dq_r}{\frac{\partial H}{\partial p_r}} = \frac{dp_s}{-\frac{\partial H}{\partial q_s}} = d\theta. \quad (r, s = 1, 2, \cdots, m)$$

Furthermore, the function H is now obviously a first integral. In fact, the partial-differential equation $X(F) = 0$, which determines the first integrals, is now

$$\sum_{r=1}^{m}\left(\frac{\partial H}{\partial p_r}\frac{\partial F}{\partial q_r} - \frac{\partial H}{\partial q_r}\frac{\partial F}{\partial p_r}\right) = 0, \qquad (99.11)$$

of which an obvious solution is $F = H$. Hence, instead of a knowledge of $2m$ first integrals, we need, in order to solve the problem completely, merely a knowledge of $2m-2$ first integrals other than the energy integral $H = C$. Hence the fact that the time variable does not enter explicitly into H reduces by *two* the order of difficulty of the problem. This is but a particular instance of what always happens when we have an ignorable coördinate, that is, a coördinate q which does not enter H explicitly. Thus, for instance, if H is free of q_1, we have at once the first integral $p_1 = C$; and we need only consider the $2m$ equations left when we omit the fraction whose numerator is dq_1. A knowledge of $2m-2$ first integrals, in addition to the known one $p_1 = C$, is needed to express the $2m$ variables $(q_2, \cdots, q_m, p_1, \cdots, p_m, t)$ in terms of the independent parameter θ. Then the remaining coördinate q_1 is determined from the equation $\frac{dq_1}{d\theta} = \frac{\partial H}{\partial p_1}$, of which the right-hand side is expressible as a function of θ alone. A simple integration, accordingly, gives q_1 as a function of θ. If the student reviews the dynamic problems he has already solved, he will observe that it was generally the presence of ignorable coördinates which made the solution possible. For instance, all natural dynamic problems with one degree of freedom require merely definite integration to solve them. When there are two degrees of freedom, an additional ignorable coördinate — that is, one other than the time coördinate which is ignorable in a natural system — suffices to reduce the difficulty of the problem to mere integrations.

The summation on the left-hand side of (99.11) is of fundamental importance in the treatment of natural dynamic systems, and it is written (F, H). It is known as Poisson's bracket. When the system is not a natural one, the equation determining the first integrals is

$$\frac{\partial F}{\partial t} + (F, H) = 0. \tag{99.12}$$

100. Methods for the determination of first integrals. We have seen that the solution of any dynamic problem rests essentially upon the determination of the proper number of independent first integrals; and that even when the requisite number of first integrals cannot be found, the knowledge of one or more first integrals materially reduces the difficulty of the problem. It remains to show how first integrals may be obtained. One method which is often applicable is connected with the theory of continuous groups. We have seen that the general solution of a system of equations of the type (99.3) is given by a set of equations of the type (99.5). For a fixed value of θ, (99.5) may be regarded as defining a transformation from the n variables y to the n variables x, the transformation reducing, when $\theta = 0$, to the identity $x_s = y_s$ $(s = 1, 2, \cdots, n)$, since the y's are the initial values of the x's. When the numeric value assigned to θ is changed, we get an entirely new transformation; and if we regard θ as changing continuously, the transformation from y to x may be said to vary continuously. If we introduce the canonical variables $(z_1, z_2, \cdots, z_n)$ of (99.10), the transformation from y to x takes a very simple form. In fact, the integration of equations (99.10) is immediately seen to be

$$z_1 = \zeta_1, \cdots, z_{n-1} = \zeta_{n-1}, z_n = \theta + \zeta_n, \tag{100.1}$$

where the ζ's are the initial values of the z's. It is at once apparent that if we follow a transformation corresponding to the value $\theta = \theta_1$ by a transformation corresponding to the value $\theta = \theta_2$, the net result is the same as the single transformation corresponding to the value $\theta_1 + \theta_2$. The transformations (99.5), whose standard form is (100.1), are accordingly said to constitute a continuous group. Let us now consider the system of equations (99.3) and the continuous group of transformations defined by a second system of equations

$$\frac{d\xi_1}{E_1} = \cdots = \frac{d\xi_n}{E_n} = d\phi. \tag{100.2}$$

The system of equations (99.3) is said to admit the group defined by the system (100.2) if the points of any integral curve of the system (99.3) are sent, by the transformations of the group defined by (100.2), into the points of an integral curve, not necessarily the same, of (99.3). For example, it is evident that every system of differential equations of the type (99.3) admits the group defined by itself. For this group sends the points of an integral curve into points of the same integral curve. In fact, the various points of an integral curve are, by definition, obtained from any one of them by varying continuously the parameter θ of the group (see (99.5)). Since the integral spreads determine and are determined by the integral curves, we may give the definition in an alternative form by saying that the system (99.3) admits the group defined by the system (100.2) when any first integral of the system (99.3) is sent by each of the transformations of the group into a first integral. On expanding any one of the transformations defined by (100.2) as a power series in the parameter ϕ, we have

$$\xi_r = \eta_r + \left(\frac{d\xi_r}{d\phi}\right)_{\phi=0}\phi + \frac{1}{2}\left(\frac{d^2\xi_r}{d\phi^2}\right)_{\phi=0}\phi^2 + \cdots, \quad (r = 1, \cdots, n)$$

where η_r denotes the value of ξ_r when $\phi = 0$. If these initial values are denoted by x_r, we have, since

$$\left(\frac{d\xi_r}{d\phi}\right) = E_r(\xi_1, \xi_2, \cdots, \xi_n),$$

$$\left(\frac{d\xi_r}{d\phi}\right)_{\phi=0} = E_r(x_1, x_2, \cdots, x_n).$$

The application of the group defined by (100.2) to any function $F(x_1, x_2, \cdots, x_n)$ amounts, then, to replacing x_r by

$$\xi_r = x_r + E_r(x_1, \cdots, x_n)\phi + \text{higher powers in } \phi.$$

For F we then have the value

$$F(\xi_1, \cdots, \xi_n) = F(x_1, \cdots, x_n) + \left(\frac{dF}{d\phi}\right)_{\phi=0}\phi + \frac{1}{2}\left(\frac{d^2F}{d\phi^2}\right)_{\phi=0}\phi^2 + \cdots.$$

Since ϕ enters F indirectly through the ξ's, we have

$$\left(\frac{dF}{d\phi}\right) = \frac{\partial F}{\partial \xi_1}\frac{d\xi_1}{d\phi} + \cdots + \frac{\partial F}{\partial \xi_n}\frac{d\xi_n}{d\phi} = E_1\frac{\partial F}{\partial \xi_1} + \cdots + E_n\frac{\partial F}{\partial \xi_n}.$$

Putting $\phi = 0$ amounts to replacing the ξ's by the x's; and so we have

$$\left(\frac{dF}{d\phi}\right)_{\phi=0} = E(F),$$

the variables in the operator E and in F being the x's. The second derivative, when $\phi = 0$, is accordingly found by applying the operator E twice, and so on. We see, then, that the result of applying any transformation of the group defined by the system (100.2) to any function $F(x_1, x_2, \cdots, x_n)$ is to replace F by the series

$$F + E(F)\phi + \tfrac{1}{2}E^2(F)\phi^2 + \cdots.$$

If F is a first integral of the system (99.3), and if the result of applying any transformation of the group defined by (100.2) to F is to be also a first integral, the various coefficients of ϕ in the power series just given must be first integrals; for ϕ can be chosen arbitrarily. As a first result we see that in order that the system (99.3) may admit the group defined by (100.2), it is necessary that if F is any first integral of (99.3), then $E(F)$ shall also be a first integral. That this condition is not only necessary but also sufficient then follows. For if the fact that F is a first integral implies that $E(F)$ is also a first integral, then the fact that $E(F)$ is a first integral implies that $E^2(F)$ is a first integral. This, in turn, implies that $E^3(F)$ is a first integral, and so on for the various repetitions of the operator E.

We see, then, that if we know beforehand that our system of equations (99.3) admits a certain group of continuous transformations, we have a method of deriving from a given first integral of our system a set of first integrals. All we have to do is to calculate $E(F)$, $E^2(F)$, and so on. Of course, we have no guarantee a priori that the first integrals so obtained are functionally independent of the one already in our possession. In fact, $E(F)$ may reduce to a mere constant, thus becoming trivial.

The test by which we know whether the system of equations (99.3) admits the group defined by the system (100.2) may be put in a form which is frequently very convenient and which is due to Poincaré. Thus, if we operate on any function F of the x's, — not necessarily a first integral of the system (99.3), — with the operator X, and follow this by an application of the operator E, we have

$$E.X(F) = E.\sum_{r} X_r \frac{\partial F}{\partial x_r} = \sum_{r,s}\left(E_s X_r \frac{\partial^2 F}{\partial x_s \partial x_r} + E_s \frac{\partial F}{\partial x_r}\frac{\partial X_r}{\partial x_s}\right).$$

Upon interchanging the rôles played by the operators X and E, and subtracting the result thus obtained from the expression just given, we obtain the identical relation

$$E.X(F) - X.E(F) = \sum_{r,s}\left(E_s \frac{\partial X_r}{\partial x_s} - X_s \frac{\partial E_r}{\partial x_s}\right)\frac{\partial F}{\partial x_r}. \quad (100.3)$$

If, now, F happens to be a first integral of the system (99.3), and if this system admits the group defined by the operator E, both $X(F)$ and $X.E(F)$ vanish identically, — the first by definition, and the second since $E(F)$ is a first integral when F is. The right-hand side of (100.3) vanishes, then, whenever F is a first integral; and so the equation obtained by setting it equal to zero is essentially the same as the equation $X(F) = 0$ which determines the first integrals. In other words,

$$\sum_{s}\left(E_s \frac{\partial X_r}{\partial x_s} - X_s \frac{\partial E_r}{\partial x_s}\right) = \rho X_r, \quad (r = 1, 2, \cdots, n) \quad (100.4)$$

where ρ is an arbitrary function of the x's. This result may be written more briefly in the form

$$E(X_r) - X(E_r) = \rho X_r. \quad (100.5)$$

A particularly simple case occurs when ρ vanishes. Then we have

$$E(X_r) = X(E_r), \quad (r = 1, 2, \cdots, n) \quad (100.6)$$

and the two operators E and X are said to be in involution. In this case it is evident that either of the two systems (99.3) and (100.2) of first-order differential equations admits the group defined by the other; for the right-hand side of (100.3) vanishes identically. Equations (100.6) are known as Poincaré's variational equations. Remembering that for any function F of the x's we have

$$\frac{dF}{d\theta}=\sum_r \frac{\partial F}{\partial x_r}\frac{dx_r}{d\theta}=\sum_r \frac{\partial F}{\partial x_r}X_r = X(F),$$

we may write the variational equations in the form

$$\frac{dE_r}{d\theta}=E(X_r). \quad (r=1, 2, \cdots, n) \qquad (100.7)$$

Now we have seen in (99.12) that the equation determining the first integrals of the canonical equations of dynamics is

$$X(F)=\frac{\partial F}{\partial t}+(F, H)=0, \qquad (100.8)$$

where (F, H) is Poisson's bracket. If, now, we differentiate the identity (100.8) with respect to any of the position coördinates (q_s, say), we find

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial q_s}\right)=\sum_r\left(\frac{\partial F}{\partial p_r}\frac{\partial^2 H}{\partial q_r \partial q_s}-\frac{\partial F}{\partial q_r}\frac{\partial^2 H}{\partial p_r \partial q_s}\right).$$

Similarly, on differentiating the identity (100.8) with respect to p_s, we obtain

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial p_s}\right)=\sum_r\left(\frac{\partial F}{\partial p_r}\frac{\partial^2 H}{\partial q_r \partial p_s}-\frac{\partial F}{\partial q_r}\frac{\partial^2 H}{\partial p_r \partial p_s}\right).$$

These equations tell us that the $2m+1$ quantities

$$E_r=\frac{\partial F}{\partial p_r}, \quad E_{m+r}=-\frac{\partial F}{\partial q_r}, \quad E_{2m+1}=0, \quad (r=1, \cdots, m) \qquad (100.9)$$

satisfy the variational equations. Hence we have the theorem that the dynamic equations admit the group defined by the equations

$$\frac{d\xi_1}{\frac{\partial F}{\partial p_1}}=\cdots=\frac{d\xi_{m+1}}{-\frac{\partial F}{\partial q_1}}=\cdots=\frac{dt}{0},$$

provided only that F is a first integral of the equations. If F_1 and F_2 are any two first integrals of the dynamic equations, we may use F_1 to construct the operator E_1, as in (100.9); and then, operating on F_2 with E_1, we obtain the new first integral

$$E_1(F_2) = \sum_r \left(\frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial q_r} - \frac{\partial F_1}{\partial q_r} \frac{\partial F_2}{\partial p_r} \right),$$

which is nothing but the Poisson bracket (F_2, F_1). This proves *Poisson's theorem*, namely,

The Poisson bracket of two first integrals of the dynamic equations is itself a first integral of these equations.

In particular, we see that if the system is a natural one, so that H is a first integral, then (F, H) is a first integral if F is. Since $\frac{\partial F}{\partial t} + (F, H) = 0$, and since (F, H), being a first integral, remains constant along any integral curve, $\frac{\partial F}{\partial t}$ remains constant along any integral curve. In other words, $\frac{\partial F}{\partial t}$ is a first integral. From this we derive, in turn, the integral $\frac{\partial^2 F}{\partial t^2}$, and so on. This result is also evident from the fact that since the time does not enter the equations of motion explicitly, the dynamic equations of a natural system admit the transformations of the group

$$q'_1 = q_1, \cdots, q'_m = q_m; \quad p'_1 = p_1, \cdots, p'_m = p_m; \quad t' = t + \phi,$$

of which the differential equations are

$$\frac{dq_1}{0} = \cdots = \frac{dp_1}{0} = \cdots = \frac{dt}{1} = d\phi.$$

The operator E defined by this group is simply $\frac{\partial}{\partial t}$; hence if F is a first integral, so also is $\frac{\partial F}{\partial t}$.

101. Integral invariants and their connection with the determination of first integrals. The system of equations (99.3) defines for each point y an integral curve having this point for its initial point, the equations of this integral curve being given in (99.5). If, instead of a single initial point, we consider a whole curve of initial points y, we shall have a whole series of integral curves forming a spread of two dimensions. Put mathematically, the numbers y are now functions of a single independent variable which we may denote by α, so that the variables x in (99.5) are now functions of two independent parameters θ and α, the latter entering through the *y's*. Denoting, for convenience, differentiation with respect to α by the symbol δ, the symbol d being reserved for differentiation with respect to the other parameter, θ, let us consider the line integral

$$I_1 = \int (A_1\,\delta x_1 + A_2\,\delta x_2 + \cdots + A_n\,\delta x_n).$$

The variable of integration is α, and the coefficients A are supposed to be functions of the x's. The other parameter, θ, will enter I_1 through the coefficients A and through the differentials $\delta x_r = \frac{\partial x_r}{\partial \alpha}\,d\alpha$. The limits of the integral, being two values of α, will not, however, vary with θ, as the two parameters α and θ are supposed to be independent. If I_1 is free from θ, so that it maintains a constant value as θ varies continuously, the integral I_1 is said to be an integral invariant of the differential equations (99.3). In order to find what conditions this imposes on the coefficients A, we calculate the differential $dI_1 = \frac{dI_1}{d\theta}\,d\theta$,

$$dI_1 = \int (A_1\,d\,\delta x_1 + \delta x_1\,dA_1 + \cdots).$$

Now, since θ enters A_1 only through the x's,

$$dA_1 = \sum_r \frac{\partial A_1}{\partial x_r}\,dx_r,$$

and $$d\,\delta x_1 = \frac{\partial^2 x_1}{\partial\theta\,\partial\alpha}\,d\theta\,d\alpha = \delta\,dx_1 = \delta X_1\,d\theta = \left(\sum_s \frac{\partial X_1}{\partial x_s}\,\delta x_s\right)d\theta.$$

On collecting terms, we see that the coefficient of any one of the differentials (δx_s, say) in the integrand of dI_1 is

$$\left(\sum_r A_r\frac{\partial X_r}{\partial x_s} + \frac{dA_s}{d\theta}\right)d\theta.$$

In order that dI_1 may be zero, no matter what the curve of integration is, — that is, independently of the values assigned to the differentials δx_s, — it is necessary and sufficient that each of the coefficients of these differentials in the integrand of dI_1 should vanish separately. This gives us the n conditions for the coefficients A,

$$\frac{dA_s}{d\theta} + \sum_r A_r\frac{\partial X_r}{\partial x_s} = 0. \quad (s = 1, 2, \cdots, n) \qquad (101.1)$$

In precisely the same way, we may consider, instead of a curve of initial points y, a series of points y lying on a spread of two dimensions. Then, starting from each of these points y, there will be an integral curve, and the whole aggregate of integral curves will fill a spread of three dimensions. Mathematically the y's will now be functions of two independent parameters α and β, and the x's, as given in (99.5), will be functions of three independent parameters θ, α, and β, the two latter entering through the y's. We may now set up an integral

$$I_2 = \int \sum_{r,s}{}' A_{rs}\,\delta(x_r, x_s),$$

where the coefficients A_{rs} are supposed to be functions of the x's. Then I_2 will be a function of the parameter θ which enters it through the coefficients A_{rs} and through the differentials δx_r. If it turns out that I_2 is free of θ, so that it maintains a constant value as θ varies continuously, the integral I_2 is said to be invariant. The conditions imposed on the coefficients are found, as before, by calculating dI_2, it being observed that the limits of the integral are independent of θ since the parameters θ, α, and β are assumed to be independent of each other. It will be sufficient to state the result:

The coefficients must satisfy the equations

$$\frac{dA_{rs}}{d\theta} + \sum_p A_{ps}\frac{\partial X_p}{\partial x_r} + \sum_p A_{rp}\frac{\partial X_p}{\partial x_s} = 0. \qquad (101.2)$$

This result may be extended to an invariant integral of any order. There will be just as many separate summations as there are labels on the coefficients of the integrand. In writing these equations it will be remembered that the coefficients of these integrals are always alternating. It will be useful to write out the criterion for invariance in the case where the order of the integral is the same as the number n of dimensions of the space, that is, when the integral is a regional one. Here there is but one coefficient $A_{123}\cdots$, and this may be denoted by the single symbol A. On account of the alternating character of the coefficient, there will be only one nonvanishing term in each of the n summations. Hence, in order that the regional integral

$$\int A\,\delta(x_1, x_2, \cdots, x_n)$$

be an invariant integral for the equations (99.3), it is necessary and sufficient that A should satisfy the equation

$$\frac{dA}{d\theta}+A\left(\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_n}{\partial x_n}\right). \tag{101.3}$$

Observing that $\frac{dA}{d\theta}=X(A)$, this may be written in the form

$$\frac{\partial}{\partial x_1}(AX_1)+\frac{\partial}{\partial x_2}(AX_2)+\cdots+\frac{\partial}{\partial x_n}(AX_n). \tag{101.4}$$

If A is regarded, for the moment, as the density of a supposed distribution of matter, this is the equation of continuity which is the mathematical expression of the principle of conservation of mass.

Before proceeding to some applications of these general results to the equations of dynamics, it may be remarked that it is frequently convenient to consider what are known as relative integral invariants. The only difference is that here the spread over which the integral is being extended must be closed. The conditions imposed on the coefficients of a relative invariant are readily found by first using Stokes's theorem to replace the given integral over a closed spread by another, but equivalent, integral over the open spread of one higher dimension bounded by the closed spread, and then applying the results already obtained. One illustration will suffice.

Thus, let us find the conditions on the coefficients A so that the integral $\oint\sum_r A_r\,\delta x_r$ may be a relative integral invariant of the equations (99.3). We have, from Stokes's theorem,

$$\oint\sum_r A_r\,\delta x_r=\int\sum_{r,s}{}' A_{rs}\,\delta(x_r,\,x_s),$$

where

$$A_{rs}=\frac{\partial A_s}{\partial x_r}-\frac{\partial A_r}{\partial x_s}.$$

Using (101.2) we find, after a little reduction, the equations

$$\frac{\partial}{\partial x_r}(X(A_s))-\frac{\partial}{\partial x_s}(X(A_r))+\sum_p\left(\frac{\partial A_p}{\partial x_r}\frac{\partial X_p}{\partial x_s}-\frac{\partial A_p}{\partial x_s}\frac{\partial X_p}{\partial x_r}\right)=0. \tag{101.5}$$

This result may be used to check the fact that the integral

$$\oint(p_1\,\delta q_1+p_2\,\delta q_2+\cdots+p_n\,\delta q_n-H\,\delta t)$$

is a relative integral invariant of the canonical equations of dynamics. It is, however, more instructive to see how this relative integral invariant was discovered.

We have seen, in (92.1), that the variation of the action integral along an integral curve of the dynamic equations, when the end points are varied, is given by the formula

$$\delta A = \left(\sum_r p_r\, \delta q_r - H\, \delta t\right)_0^1.$$

If we take the beginning point around any closed curve, the final point, corresponding to a given value of θ, will also trace out a closed curve, and the path along which the action integral is extended will return to its original position. Hence the integral $\oint \delta A$ will be zero, showing that the integral $\oint \left(\sum_r p_r\, \delta q_r - H\, \delta t\right)$ has the same value when taken around the closed curve traced out by the end point as it has when taken around the closed curve traced out by the initial point. Since the value of θ determining the end point may be chosen arbitrarily, this shows that the integral is a relative invariant. The integral of order two, into which it transforms by an application of Stokes's theorem, and which is therefore an absolute integral invariant of the dynamic equations, is

$$\int \sum_r \left[\delta(q_r, p_r) + \frac{\partial H}{\partial q_r}\, \delta(q_r, t) + \frac{\partial H}{\partial p_r}\, \delta(p_r, t)\right]. \qquad (101.6)$$

When the system is a natural one, we may drop the time coördinate from consideration, and we have the relative integral invariant

$$\oint \sum_r p_r\, \delta q_r, \qquad (101.7)$$

the corresponding absolute invariant being

$$\int \sum_r \delta(q_r, p_r). \qquad (101.8)$$

It may be remarked here that when the canonical variables of (99.10) are available, the conditions imposed on the coefficients of an integral in order that it may be an integral invariant take an extremely simple form. Since all the Z's are constants, equations (101.1) reduce to $\dfrac{dA_s}{d\theta} = 0$; that is, $\dfrac{\partial A_s}{\partial z_n} = 0$. In other words, a line integral of order one is invariant when its coefficients are all free of the variable coördinate z_n. For the same reason, this result holds for an integral of any order. The importance of this result is that it assures us that if we are able to build up from two integrals a third integral whose coefficients are functions of the coefficients of the first two, the new integral will be invariant if the two original integrals are invariant. For the coefficients of the new integral will be free of the variable coördinate z_n if the coefficients of the two integrals from which it is formed are free of z_n. Now there is an operation called outer multiplication which enables us to construct from two given integrals, of orders p and q respectively, a new integral of order $p + q$, which is called the outer product of the other two. We shall merely state the rule (for a proof see *Bulletin of the American Mathematical Society*, Vol. XXXI (1925), pp. 323-329). If the coefficients of the two integrals are denoted by A and B respectively, with the appropriate suffixes, the coefficients C of the outer product are given by the rule

$$C_{r_1, r_2, \cdots, r_{p+q}} = \Sigma A_{m_1, m_2, \cdots, m_p} B_{n_1, n_2, \cdots, n_q}. \tag{101.9}$$

In the summation on the right, $(m_1, m_2, \cdots, m_p)$ is any group of p out of the $p + q$ numbers $(r_1, r_2, \cdots, r_{p+q})$; and $(n_1, n_2, \cdots, n_q)$ are the remaining q numbers arranged so that it requires an even number of transpositions to get from the arrangement $(m_1, m_2, \cdots, m_p, n_1, n_2, \cdots, n_q)$ to the arrangement $(r_1, r_2, \cdots, r_{p+q})$. If arranged otherwise, the sign before the term must be negative instead of positive. For example, if $p = 1$ and $q = 1$, the outer product of the two line integrals $\int \sum_r A_r \, \delta x_r$ and $\int \sum_r B_r \, \delta x_r$ is the double integral $\int \sum'_{r,s} C_{rs} \, \delta(x_r, x_s)$, where $C_{rs} = A_r B_s - A_s B_r$. The negative sign before the second term is necessary here, since the arrangement sr requires an odd number (one) of transpositions to bring it to the arrangement rs. An application of this theorem of outer multiplication of integrals to the case of the integral invariant $\int \sum_r \delta(q_r, p_r)$ of a natural dynamic system gives us, on forming the outer product of this integral by itself, the integral invariant of order four,

$$\int \sum'_{r,s} \delta(q_r, q_s, p_r, p_s). \tag{101.10}$$

Taking the outer product of this, in turn, by the original integral invariant (101.8), we get the integral invariant of order six $\int\sum_{r,s,t}'\delta(q_r, q_s, q_t, p_r, p_s, p_t)$. Continuing this process we finally arrive at the regional integral of order 2 m,

$$\int\delta(q_1, \cdots, q_m, p_1, \cdots, p_m). \tag{101.11}$$

This is an invariant integral of the equations of a natural dynamic system, the representative space being the phase space of 2 m dimensions. This result may be checked by showing that any constant, in particular unity, is a solution of (101.4) for a natural dynamic system; for then $n = 2m$, the determination of the time being postponed until the rest of the problem is solved, and the terms $\frac{\partial}{\partial x_r}(X_r)$ and $\frac{\partial}{\partial x_{m+r}}(X_{m+r})$ mutually cancel.

The solutions A of equation (101.4) are known as *multipliers* of the set of differential equations (99.3), for the following reason. Suppose that although the complete solution of the problem — that is, the knowledge of $n - 1$ functionally distinct first integrals — is not in our possession, we do know $n - 2$ functionally distinct first integrals. On introducing the variables z defined by the equations

$$F_1 = z_1, \quad \cdots, \quad F_{n-2} = z_{n-2},$$

where the F's are the known first integrals, and using these to eliminate $n - 2$ of the x's, — $(x_1, x_2, \cdots, x_{n-2})$, say, — as explained in § 99, our equations take the simple form

$$\frac{dz_1}{0} = \cdots = \frac{dz_{n-2}}{0} = \frac{dx_{n-1}}{Z_{n-1}} = \frac{dx_n}{Z_n} = d\theta,$$

where Z_{n-1} and Z_n stand for X_{n-1} and X_n when expressed in terms of the $n - 2$ constants $(z_1, z_2, \cdots, z_{n-2})$ and the two variables (x_{n-1}, x_n). The complete integration of the differential equations requires, then, merely an integration of the equation

$$Z_n dx_{n-1} - Z_{n-1} dx_n = 0. \tag{101.12}$$

This can be done if we know an integrating factor, that is, a function μ of x_{n-1} and x_n satisfying the equation

$$\frac{\partial}{\partial x_n}(\mu Z_n) + \frac{\partial}{\partial x_{n-1}}(\mu Z_{n-1}) = 0. \tag{101.13}$$

In fact, if this equation is satisfied, μZ_n and $-\mu Z_{n-1}$ are, respectively, the derivatives with respect to x_{n-1} and x_n of a function F of the two variables x_{n-1} and x_n and the $n - 2$ constants $(z_1, z_2, \cdots, z_{n-2})$.

Equation (101.12) then yields, on integration, the first integral $F =$ constant. Now equation (101.13) is exactly what equation (101.4) becomes when the variables $(z_1, z_2, \cdots, z_{n-2}, x_{n-1}, x_n)$ are introduced, since then the denominators $(Z_1, Z_2, \cdots, Z_{n-2})$ are all zero.

It remains to be seen what connection there is between a solution A of equation (101.4) and a solution μ of the similar equation (101.13) in the new variables. This connection follows at once from the geometric significance of the solutions A of (101.4), namely, that the regional integral $\int A\delta(x_1, x_2, \cdots, x_n)$ is an integral invariant of equations (99.3). When a change of variables from x to z is made, this integral takes the form $\int \mu\delta(z_1, z_2, \cdots, z_n)$, where

$$\mu = A\,\frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(z_1, z_2, \cdots, z_n)}.$$

Since $\int \mu\delta(z_1, z_2, \cdots, z_n)$ is an integral invariant of the transformed equations

$$\frac{dz_1}{z_1} = \frac{dz_2}{z_2} = \cdots = \frac{dz_n}{z_n} = d\theta,$$

μ will be a solution of the equation

$$\frac{\partial}{\partial z_1}(\mu Z_1) + \cdots + \frac{\partial}{\partial z_n}(\mu Z_n) = 0.$$

Applying this remark to the particular case in hand, we see that by multiplying any solution of (101.4) by the Jacobian determinant $\dfrac{\partial(x_1, x_2, \cdots, x_{n-2})}{\partial(z_1, z_2, \cdots, z_{n-2})}$ we have a solution of equation (101.13). Calling any solution of (101.4) a multiplier, we see that the knowledge of $n - 2$ functionally independent first integrals and a multiplier suffices to complete the integration of equations (99.3). It is at once evident, from (101.3), that the quotient of two multipliers M_1 and M_2 is a first integral; for we find

$$M_2\frac{dM_1}{d\theta} - M_1\frac{dM_2}{d\theta} = 0, \quad \text{or} \quad \frac{d}{d\theta}\left(\frac{M_1}{M_2}\right) = 0,$$

which shows that $\dfrac{M_1}{M_2}$ remains constant along an integral curve.

These results have an immediate application to the problem of integrating the canonical equations of dynamics. Since now

$$X_1 = \frac{\partial H}{\partial p_1}, \quad \cdots, \quad X_{m+1} = -\frac{\partial H}{\partial p_1}, \quad \cdots, \quad X_{2m+1} = 1,$$

equation (101.3), determining the multipliers, is simply $\dfrac{dA}{d\theta} = 0$, so that the multipliers and first integrals are identical. This does not, however, make our results vacuous, since the trivial first-integral unity is not trivial when regarded as a multiplier. The knowledge of this multiplier assures us that a knowledge of $2m - 1$, instead of $2m$, first integrals suffices to solve the problem. When the system is a natural one, the ignorable coördinate t reduces by two units the order of the problem, and a knowledge of $2m - 3$ first integrals, functionally independent and distinct from the energy integral, solves the problem. Thus any problem with two degrees of freedom, the system being a natural one, can be solved if one integral in addition to the energy integral is known. The fact that unity is a multiplier for the dynamic equations may be expressed in a rather striking form as follows:

If we regard each state of the dynamic system as represented by a particle in the representative state (or phase) space, these particles may be thought of as moving like the molecules of an incompressible fluid.

The coördinates are assumed to be rectangular Cartesian coördinates in the representative space.

102. Application of our general results to the problem of three bodies. In order to illustrate the general theorems obtained in the preceding paragraphs, we shall consider the problem of three bodies moving in the field of force determined by their mutual attractions. The potential energy of the system is $-\mu\sum_{i,j}\frac{m_i m_j}{r_{ij}}$, there being six terms in the summation. The system has nine degrees of freedom, there being three positional coördinates for each particle. Denoting the rectangular Cartesian coördinates of the ith particle by $(q_{3i-2}, q_{3i-1}, q_{3i})$, and the components of the linear momentum of this particle by $(p_{3i-2}, p_{3i-1}, p_{3i})$, where $i = 1, 2$, or 3, the kinetic energy of the system, when expressed in terms of the momenta, is $\frac{1}{2}\sum_r \frac{p_r^2}{\overline{m}_r}$, where $\overline{m}_{3i-2} = \overline{m}_{3i-1} = \overline{m}_{3i} = m_i$ is the mass of the ith particle. The system is a natural one, with the Hamiltonian function

$$H = \frac{1}{2}\sum_r \frac{p_r^2}{\overline{m}_r} - \mu\sum_{i,j}\frac{m_i m_j}{r_{ij}} \tag{102.1}$$

Since time is an ignorable coördinate, we postpone its determination and consider only the remaining $2m = 18$ equations, of which H is a known first integral. Since unity is a multiplier of the equations, a knowledge of $2m - 2 = 16$ first integrals other than the energy integral would suffice to solve the problem.

A number of first integrals are evident from elementary considerations; but as it is our purpose here to illustrate the general theory rather than to arrive at the results in this special problem in the quickest way, we shall not write these integrals down at once. Considering the equations

$$\frac{dq_1}{\dfrac{\partial H}{\partial p_1}} = \cdots = \frac{dp_1}{-\dfrac{\partial H}{\partial q_1}} = \cdots = d\theta, \tag{102.2}$$

there are certain groups which the nature of the problem shows that these equations must admit. For instance, if we displace the origin of measurement parallel to the x-axis, none of the p's nor the r_{ij} will be affected; and so equations (102.2) will be unaltered. In other words, if the whole configuration of the three particles is moved a certain distance parallel to the x-axis, the subsequent motion relative to a set of axes obtained by displacing the original set the same distance parallel to the x-axis will be the same as the motion of the undisplaced system with respect to the original axes. Hence the group of transformations

$$\xi_1 = q_1 + \phi, \quad \xi_4 = q_4 + \phi, \quad \xi_7 = q_7 + \phi$$

(all other ξ's being equal to the corresponding q's or p's) sends an integral curve of equations (102.2) into an integral curve of these equations. For this group we have $E_1 = \dfrac{d\xi_1}{d\phi} = 1$ etc., so that the operator of the group is

$$\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_4} + \frac{\partial}{\partial q_7}.$$

Calling this group the first group admitted by the equations, we have

$$E_1(F) = \frac{\partial F}{\partial q_1} + \frac{\partial F}{\partial q_4} + \frac{\partial F}{\partial q_7}. \tag{102.3}$$

Our general theory tells us, then, that $E_1(H)$ must be a first integral; but we find, on performing the calculation, that this result is vacuous, since $E_1(H)$ vanishes identically. By considering, in turn, the groups of translations parallel to the y and z axes respectively, we find two other groups admitted by equations (102.2). Their operators are

$$E_2(F) = \frac{\partial F}{\partial q_2} + \frac{\partial F}{\partial q_5} + \frac{\partial F}{\partial q_8} \tag{102.4}$$

and

$$E_3(F) = \frac{\partial F}{\partial q_3} + \frac{\partial F}{\partial q_6} + \frac{\partial F}{\partial q_9}. \tag{102.5}$$

In addition to the groups of translations, equations (102.2) admit, for the same reasons, the groups of rotations. Thus, if we consider a rotation of angle α about the x-axis, its equations are

$$\begin{aligned} \xi_{3i-1} &= q_{3i-1}\cos\alpha + q_{3i}\sin\alpha, \\ \xi_{3i} &= -q_{3i-1}\sin\alpha + q_{3i}\cos\alpha, \end{aligned} \qquad (i = 1, 2, 3)$$

and six similar equations for the momenta. The remaining three position coördinates and their corresponding momenta are unaltered. Putting $\alpha = 0$, we get the identical transformation; and so the coefficients of the operator are found by calculating $\frac{d\xi_{3i-1}}{d\alpha}$ etc. and then putting $\alpha = 0$. Calling this group the fourth group admitted by equations (102.2), its operator E_4 is found in this way to be

$$E_4(F) = \sum_i \left(q_{3i}\frac{\partial F}{\partial q_{3i-1}} - q_{3i-1}\frac{\partial F}{\partial q_{3i}} + p_{3i}\frac{\partial F}{\partial p_{3i-1}} - p_{3i-1}\frac{\partial F}{\partial p_{3i}} \right). \qquad (102.6)$$

Applying this operator to the energy integral H, we find that $E_4(H)$ vanishes identically, so that it does not furnish us a new first integral. By considering the groups of rotations about the y and z axes respectively, we have two other groups admitted by equations (102.2), their operators being

$$E_5(F) = \sum_i \left(q_{3i-2}\frac{\partial F}{\partial q_{3i}} - q_{3i}\frac{\partial F}{\partial q_{3i-2}} + p_{3i-2}\frac{\partial F}{\partial p_{3i}} - p_{3i}\frac{\partial F}{\partial p_{3i-2}} \right), \qquad (102.7)$$

and

$$E_6(F) = \sum_i \left(q_{3i-1}\frac{\partial F}{\partial q_{3i-2}} - q_{3i-2}\frac{\partial F}{\partial q_{3i-1}} + p_{3i-1}\frac{\partial F}{\partial p_{3i-2}} - p_{3i-2}\frac{\partial F}{\partial p_{3i-1}} \right). \qquad (102.8)$$

These six groups constitute, when combined, the entire group of rigid displacements in space. Any dynamic system in which the forces are merely functions of the relative distances of any two particles of the system will admit this group of rigid displacements. In order to find other groups admitted by our dynamic equations, we shall consider the extended system in which the time is included,

$$\frac{dq_1}{\dfrac{\partial H}{\partial p_1}} = \cdots = \frac{dp_1}{-\dfrac{\partial H}{\partial q_1}} = \cdots = dt = d\theta. \tag{102.9}$$

Now we know, from the Galilean or Newtonian principle of relativity, that if we impress on the system of three particles a uniform velocity in a given direction, the motion relative to a reference frame having this velocity relative to the original frame will be the same as the motion of the undisturbed system relative to the original frame (see § 34). Let us consider the group of uniform velocities parallel to the x-axis. The equations of this group are

$$\xi_1 = q_1 + t\phi, \quad \xi_4 = q_4 + t\phi, \quad \xi_7 = q_7 + t\phi$$

(with similar equations $\eta_1 = p_1 + \overline{m}_1\phi$ etc. for the corresponding momenta), the other position coördinates and momenta and t not being altered. The uniform velocity is ϕ, and the identical transformation is found by putting $\phi = 0$. The operator of the group is found, as before, by calculating the derivatives $\dfrac{d\xi}{d\phi}$ and putting $\phi = 0$. Calling this group the seventh group admitted by equations (102.9), we have

$$E_7(F) = \sum_i \left(t\frac{\partial F}{\partial q_{3i-2}} + \overline{m}_{3i-2}\frac{\partial F}{\partial p_{3i-2}} \right). \tag{102.10}$$

Similarly, by considering the groups of uniform velocities parallel to the y and z axes respectively, we have two other groups admitted by equations (102.9). Their operators are

$$E_8(F) = \sum_i \left(t\frac{\partial F}{\partial q_{3i-1}} + \overline{m}_{3i-1}\frac{\partial F}{\partial p_{3i-1}} \right) \tag{102.11}$$

and

$$E_9(F) = \sum_i \left(t\frac{\partial F}{\partial q_{3i}} + \overline{m}_{3i}\frac{\partial F}{\partial p_{3i}} \right). \tag{102.12}$$

If we apply the operator E_7 to H, we get the new first integral

$$\sum_i p_{3i-2}.$$

Similarly, on operating with E_8 and E_9, we obtain two other first integrals. These are simply the momentum integrals expressing the fact that the total linear momentum of the system is constant. Denoting them by F_1, F_2, and F_3, we have

$$F_1 = p_1 + p_4 + p_7, \quad F_2 = p_2 + p_5 + p_8, \quad F_3 = p_3 + p_6 + p_9. \tag{102.13}$$

We may now proceed methodically to apply each of our nine operators to each of these three new first integrals, with a view to obtaining others. We find, however, that we do not get any new integrals in this way, the results of the operations being either numeric constants or multiples of the integrals already in our possession. However, on observing that $p_1 = m_1 \dfrac{dq_1}{dt}$ etc., we obtain, on integrating F_1, the new first integral

$$F_4 = m_1 q_1 + m_2 q_4 + m_3 q_7 - (p_1 + p_4 + p_7)t. \tag{102.14}$$

Similarly, we may obtain the other two integrals

$$F_5 = m_1 q_2 + m_2 q_5 + m_3 q_8 - (p_2 + p_5 + p_8)t \tag{102.15}$$

and

$$F_6 = m_1 q_3 + m_2 q_6 + m_3 q_9 - (p_3 + p_6 + p_9)t. \tag{102.16}$$

It may be observed, in passing, that F_5 could have been obtained from F_4 by applying to it the operator E_6.

We have indicated sufficiently the general method of search for first integrals; but the reader will, no doubt, have realized that the method outlined above, while systematic, is not sufficiently powerful. For instance, the apparently fortuitous manner in which F_4 was derived from F_1 leaves one with the feeling that its derivation was a piece of luck peculiar to this problem. We shall therefore indicate a method of search for first integrals which is of general application, but without explaining all the details.

We have seen, in § 101, that when the canonical variables $(z_1, z_2, \cdots, z_n)$ of (99.10) are introduced, the necessary and sufficient condition for an integral over a spread of any number of dimensions to be an invariant integral of the equations (99.3) is that none of the coefficients may involve the variable z_n. We saw also, in § 100, that the necessary and sufficient condition for the operator E to be in involution with the operator X is that none of the coefficients E involve the variable z_n. If, then, we take the inner product of the operator E, supposing it to be in involution with the operator X, and the coefficients of a line integral, — that is, if we form the sum of products $\sum E_r A_r$, — we shall have a function which is free from z_n, provided the line integral is an invariant one. In other words, the function

$$F = \sum_r E_r A_r, \tag{102.17}$$

arrived at in this way, will be a first integral of the equations (99.3). In particular, the function

$$F = \sum_r X_r A_r \tag{102.18}$$

is a first integral (a result due to Poincaré). Similarly, if we have an invariant integral of order two, with coefficients A_{rs}, and form the inner product

$$C_r = \sum_s E_s A_{rs},$$

these will be the coefficients of an invariant line integral. On operating upon this invariant line integral as just explained, using any operator which is in involution with X, we obtain a first integral of the equations (99.3). In the dynamic problem we always have the integral invariant of order two given in (101.6). The operator E_1 is seen to be in involution with X, and the inner product of this with (101.6) gives us the integral invariant of order one,

$$\int \delta p_1 + \delta p_4 + \delta p_7. \tag{102.19}$$

On taking the inner product of (102.19) and the operator E_5, we obtain the integral F_3. We may refer the reader, if he is interested in this method of obtaining systematically the first integrals of a dynamic system, to M. Cartan's "Leçons sur les invariants intégraux" (Paris, 1922). It may be remarked, in conclusion, that the integrals obtainable are ten in number, namely, the energy integral, the six linear-momentum integrals already given, and three angular-momentum integrals. By a proper choice of coördinates it is possible to get an ignorable coördinate, so that the order of the problem is reducible from nineteen to six, as follows: a reduction of four units, on account of the ignorable time coördinate and the ignorable position coördinate; a reduction of nine units, on account of the nine independent first integrals other than the energy integral. It has been shown that the problem has no other algebraic first integrals.

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EXERCISES

1. Show that the Poisson-bracket expressions defined in (99.11) and (99.12) satisfy the equations

$$(F, G) = -(G, F), \quad (F, F) = 0, \quad (F, C) = 0. \quad (C = \text{constant})$$

2. If F and G are functions of the variables (q, p) indirectly through certain functions $(x_1, x_2, \cdots, x_n)$ of these variables, show that

$$(F, G) = \sum_{r\,s}{}' (x_r, x_s) \frac{\partial(F, G)}{\partial(x_r, x_s)}.$$

3. As a particular case of Exercise 2, show that

$$(F, x_n) = \sum_r (x_r, x_n) \frac{\partial F}{\partial x_r}.$$

4. Show that $(F_1 + F_2, G) = (F_1, G) + (F_2, G)$.

5. Show that $(F_1 F_2, G) = F_2(F_1, G) + F_1(F_2, G)$.

6. Show that if A, B, C are any three functions of the variables (q, p), and if we denote the Poisson brackets (B, C), (C, A), and (A, B) by A', B', and C' respectively, then

$$(A, A') + (B, B') + (C, C') = 0.$$

7. Using the fact that if F is a first integral of the dynamic equations, the line integral $\int dF$ is an integral invariant, show how to derive from a given number of first integrals an integral invariant of order equal to the number of integrals.

8. Let $H = q_1 p_1 - q_2 p_2 - a q_1{}^2 + b q_2{}^2$ be the Hamiltonian function of a dynamic system with two degrees of freedom. Show that $\frac{p_2 - b q_2}{q_1}$ is a first integral. Also show that $q_1 q_2$ and $q_1 e^{-t}$ are first integrals.

9. Using the integral invariant, $\int \sum_r \delta(q_r, p_r)$, for a natural dynamic system, state and prove Helmholtz's reciprocal theorem. The surface of integration is so chosen that initially the derivatives of all but one of the phase coördinates (q, p) with respect to one of the parameters on the surface vanish, while finally the derivatives of all but one of the phase coördinates with respect to the other parameter on the surface vanish.

10. Show that a system of equations of the form (99.3), for which $n = 2m$, and admitting the relative integral invariant $\oint \sum_1^m x_{m+r}\, \delta x_r$, is of the type furnished by the canonical equations of a natural dynamic system.

11. Show how to deduce a first integral from an integral invariant of order one, whose integrand is an exact differential δF. Apply this method to the invariant integral $\int \delta p_1 + \delta p_4 + \delta p_7$ in the problem of three bodies.

CHAPTER XVI

THE POTENTIAL FUNCTION

103. The volume potential. If we have a particle of mass m situated at the point (ξ, η, ζ) and attracting according to the inverse-square law, the force exerted by it on a particle of unit mass situated at the point (x, y, z) is the negative gradient of the function V defined by

$$V = -\frac{\mu m}{r},$$

where μ is the gravitational constant and r is the distance from the point (ξ, η, ζ) to the point (x, y, z). Similarly, if we have a series of particles of masses m_i situated at the points (ξ_i, η_i, ζ_i), the force exerted by them on a particle of unit mass situated at the point (x, y, z) is the negative gradient of the function

$$V = -\mu \sum_i \frac{m_i}{r_i}. \tag{103.1}$$

This function V measures the potential energy of the unit particle at the point (x, y, z) in the field of force of the attracting particles. It is defined at all points of the field save the points occupied by the attracting particles, and it satisfies the equation

$$\Delta_2 V = 0 \tag{103.2}$$

at all points where it is defined. Furthermore, it has the property that it tends to zero as the point (x, y, z) moves off to a remote distance from all the attracting particles. Indeed we may say, more definitely, that as r tends to infinity the limit of the product rV equals $-\mu M$, where M is the total mass of the attracting particles. Here r is the distance of the point (x, y, z) from any convenient origin; and as r tends to infinity each of the ratios $\frac{r}{r_i}$ tends to unity.

If, now, we pass from a series of discrete particles to a continuous distribution of matter throughout a volume, we find that the force of attraction of a volume distribution of density ρ

on a particle of unit mass at (x, y, z) is the negative gradient of the function

$$V = -\mu \int \frac{\rho \, d\tau}{r}, \tag{103.3}$$

where r is the distance from the point of evaluation (x, y, z) to the variable point of integration (ξ, η, ζ), and $d\tau = d(\xi, \eta, \zeta)$ is the element of volume. This function V is defined at all points save those occupied by the attracting matter, and at the points where it is defined it satisfies the equation

$$\Delta_2 V = 0. \tag{103.4}$$

In addition it vanishes at infinity in such a way that the product rV tends to the definite limit $-\mu M$, where M is the total mass of the attracting matter. The components of attraction on the unit mass at the point (x, y, z) are found, on taking the negative gradient of (103.3), to be

$$X = \int \frac{\rho(\xi - x)}{r^3} \, d\tau, \text{ etc.}, \tag{103.5}$$

and, again, these are defined at all points save those occupied by the attracting matter. Now it is inconvenient to leave the potential function, and the resulting force components, undefined at points of the attracting mass, and the method of extending the region of definition so as to include the points of the attracting mass is as follows. Let P be any point either in the interior or on the surface of the attracting mass, and let us imagine a cavity δ made in the attracting mass so that the point P and points in its immediate neighborhood are removed from the attracting mass. The function V will now be defined at the point P; but it will be the potential V_δ not of the original mass but of what this mass becomes when the cavity δ is made in it. We can now visualize a whole series of cavities $\delta_1, \delta_2, \cdots, \delta_n$, decreasing steadily in volume; and we shall have a corresponding series of potential functions $V_{\delta_1}, V_{\delta_2}, \cdots, V_{\delta_n}$, all defined at the point P. If this sequence of potential functions approaches a definite limit V as δ_n tends to zero in volume, we may call this function V of the point $P = (x, y, z)$ the potential of the original attracting mass at the point P. It must be borne in mind, however, that with this definition the potential V is not a simple integral but an improper integral, that is, the limit of an integral. The attraction components (X, Y, Z) are defined in the same way. For example, X is the limit of X_{δ_n}, so that (X, Y, Z) are really defined independently of V. The question then arises as to whether or not the vector (X, Y, Z) is the negative gradient of V. If the answer to this question were in the negative, we should give up the definition of the potential function just given as useless; for the rôle of the potential function is to furnish by its negative gradient the attraction components.

In order to prove the existence of the two limits Lt V_{δ_n} and Lt X_{δ_n}, and also that these two limits satisfy the equation $X = -\frac{\partial V}{\partial x}$, let us first consider any volume v and a point P external to it. Denoting by r the distance from a variable point (ξ, η, ζ) of the volume v to the point P, it is evident that both the integrals $\int \frac{d\tau}{r}$ and $\int \frac{d\tau}{r^2}$ are less when extended over the volume v than when extended over a sphere of the same volume having P as its center. The latter integrals are necessarily improper integrals, but their existence and, in fact, their actual values follow from the fact that $d\tau = r^2 d\omega\, dr$, where $d\omega$ is the element of solid angle at the point P. The first integral, $\int \frac{d\tau}{r}$, when extended over a sphere of radius a about the point P, has the value $2\,\pi a^2$; and the second integral, $\int \frac{d\tau}{r^2}$, has the value $4\,\pi a$. The volume of the sphere being v, we have $a = \left(\frac{3\,v}{4\,\pi}\right)^{\frac{1}{3}}$. When the point P is internal to the volume v, the same argument holds; only now the integrals $\int \frac{d\tau}{r}$ and $\int \frac{d\tau}{r^2}$ are improper. We have, then, the results

$$\begin{aligned} \int \frac{d\tau}{r} &< 2\,\pi \left(\frac{3\,v}{4\,\pi}\right)^{\frac{2}{3}}, \\ \int \frac{d\tau}{r^2} &< 4\,\pi \left(\frac{3\,v}{4\,\pi}\right)^{\frac{1}{3}}, \end{aligned} \tag{103.6}$$

where v is the volume of the region over which the integration is being extended. Thus, if we are dealing with the potential function V, we have

$$V_{\delta'_n} - V_{\delta_n} < \epsilon, \tag{103.7}$$

where ϵ depends only on the difference in volume between the two cavities δ'_n and δ_n and tends to zero with this difference. It is necessary only to assume that the density ρ is bounded in the region of integration. If it is definitely less than M, we may set $V < M \int \frac{d\tau}{r}$, thus arriving at (103.7). Now the important thing to notice is that the ϵ in (103.7) is quite independent of the position (x, y, z) of the point P, — a fact which is expressed by the statement that the convergence of V_{δ_n} to its limit V is uniform. This assures us that the limit V is a continuous function of (x, y, z). The same argument shows us that X_{δ_n} converges to a continuous function X of (x, y, z); and it is this function which is called the x component of the force of attraction of the volume distribution at the internal point P. In order to prove that $X = -\frac{\partial V}{\partial x}$, let us consider a cavity in the shape of a circular cylinder with its axis parallel to the x-axis and passing through the point (x, y, z). The proof given above shows that the existence of the limits V and X depends merely on the infinitesimal nature of the volume of the cavity δ_n and does not demand that each dimension of this cavity should be infinitesimal. Let us avail ourselves of this fact to keep the length of the cylinder unity, say, the volume being made infinitesimal by making the radius of the cylinder infinitesimal. Then, for any point on the axis of the cavity, we have $X_{\delta_n} = -\frac{\partial V_{\delta_n}}{\partial x}$, both of the integrals being proper. Integrating this along the axis of the cavity, we have $\int_{x_1}^{x_2} X_{\delta_n}\,dx = V_{\delta_n}(x_1, y, z) - V_{\delta_n}(x_2, y, z)$, where x_1 and x_2 are any two points on the axis of the cavity. These points are external to the region of integration for V_{δ_n} and X_{δ_n}, and may remain fixed as δ_n tends to zero in volume. On taking δ_n sufficiently small, we may make the difference between X_{δ_n} and X and between V_{δ_n} and V as small as we wish, the degree of the approximation not depending on the point of evaluation but only on the size of the cavity δ_n. This tells us that the expression $\int_{x_1}^{x_2} X\,dx - V(x_1, y, z) + V(x_2, y, z)$ can be made as small as we like by merely making δ_n sufficiently small. But the value of this expression is quite independent of the size of δ_n; and as this value is smaller than any assignable number, it must be zero. We have, then, the relation

$$\int_{x_1}^{x_2} X\,dx = V(x_1, y, z) - V(x_2, y, z),$$

which yields, on differentiation with respect to the upper limit x_2, the result $X(x_2, y, z) = -\frac{\partial V(x_2, y, z)}{\partial x_2}$, or, since x_2 is any point,

$$X = -\frac{\partial V}{\partial x}. \tag{103.8}$$

Since X is continuous, this proves the existence and continuity of the gradient of V at an internal point.

104. The simple-layer and double-layer potential. Instead of a volume distribution of matter, it is often convenient to consider a surface distribution. The potential function is then

$$V = -\mu \int \frac{\sigma\,dS}{r}, \tag{104.1}$$

where σ is the surface density and dS is the element of area of the surface. This function is defined at all points save those occupied by the attracting matter, and satisfies Laplace's equation (103.2) at all points where it is defined. It vanishes at infinity in such a way that the product rV tends to $-\mu M$, where M is the total mass of the attracting matter. In order to deal with points P on the surface itself, we have to imagine a sequence of cavities δ_n, each of which includes the point P, and then consider the question of the existence of $\text{Lt}\, V_{\delta_n}$ as the area of the cavity δ_n tends to zero. If we project the area of this cavity on the plane tangent to the surface at P and assume the density bounded, we find first that the integral (improper) over the cavity δ_n is less than a fixed multiple of the integral $\int \frac{dS'}{r'}$ extended over the projection of the cavity on the tangent plane. For each r' is less than the corresponding r; and, on the assumption that the direction of the normal varies continuously around P, we can find a fixed number N such that $dS' > N\,dS$ all over the cavity (N being, in fact, the least value of the cosine of the angle between the variable normal and the normal at P). Now the integral $\int \frac{dS'}{r'}$ is less than the same integral extended over a circle of the same area with its center at P; and if a is the radius of this circle, the value of the integral over the circle is $2\pi a$. An argument exactly similar to that given for the volume potential then shows that the limit $V = \text{Lt}\, V_{\delta_n}$ exists, and that V is a continuous function of (x, y, z), the convergence to the limit depending only on the size of the cavity, and not on the point of evaluation. It is worthy of notice that since the improper integral $\int \frac{dS'}{r'^2}$ does not exist for the circle, we cannot derive, as we did in the case of the volume potential, the existence of the force components (X, Y, Z) at a point of the surface. In order to examine the force components it will be necessary to say a word about what is known as the double-layer potential.

The double-layer potential is an integral of the form

$$W = \int m \frac{\partial}{\partial \nu} \frac{1}{r}\, dS, \tag{104.2}$$

where m is known as the strength of the layer and is, in general, a function of the variable point of integration, and where ν

denotes the normal to the surface (its direction being named by saying that it goes *from* the negative side of the surface *to* the positive side). The quantity W is defined at all points not on the surface of integration, but it requires a little investigation to see whether it can be defined at points on the surface or not. To carry this out we shall avail ourselves of a form of Green's lemma.

Starting from the fundamental result

$$\int \frac{\partial u}{\partial x}\, d\tau = \int u\, d(y, z) = \int lu\, dS,$$

we set

$$u = \phi \frac{\partial}{\partial x} \frac{1}{r},$$

so that

$$\int \left[\frac{\partial \phi}{\partial x} \frac{\partial}{\partial x} \frac{1}{r} + \phi \frac{\partial^2}{\partial x^2} \frac{1}{r}\right] d\tau = \int \phi l \frac{\partial}{\partial x} \frac{1}{r}\, dS.$$

On forming the similar equations for the other two axes and adding them to this one, we get

$$\int \left(\mathbf{grad}\, \phi . \mathbf{grad}\, \frac{1}{r}\right) d\tau = \int \phi \frac{\partial}{\partial \nu} \frac{1}{r}\, dS,$$

since $\Delta_2 \frac{1}{r} = 0$. If, however, the point of evaluation $P = (x, y, z)$ is an internal point, we shall have to introduce a cavity which will have both a volume and a surface and which, for the sake of convenience, may be taken as spherical. From our result on the force components of a volume distribution we see that the integral over the volume of the cavity vanishes in the limit. As for the integral over the surface of the cavity, we have

$$\int_{\delta_n} \phi \frac{\partial}{\partial \nu} \frac{1}{r}\, dS = \bar{\phi} \int_{\delta_n} \frac{\partial}{\partial \nu} \frac{1}{r}\, dS,$$

where $\bar{\phi}$ is a mean value of ϕ over the surface of δ_n. This has the value $\bar{\phi} \int_{\delta_n} d\omega$, where $d\omega$ is the element of solid angle at the center of the spherical cavity $\left(\text{the normal } \nu \text{ being directed toward the center of the cavity, we have } \frac{\partial}{\partial \nu} = -\frac{\partial}{\partial r}\right)$. If, then, the point P is an internal point, the integral over the surface of the cavity tends to the limit $4\pi\phi(P)$ as the size of the cavity is decreased indefinitely, since the limit of the mean value $\bar{\phi}$ is $\phi(P)$. If the point P is on the surface of the volume of integration, the integral over the surface of the cavity has the limit $2\pi\phi(P)$, since the surface of the cavity is now hemispherical instead of spherical. We have, then, the general result

$$\int\left(\operatorname{grad}\phi\cdot\operatorname{grad}\frac{1}{r}\right)d\tau=\int\phi\frac{\partial}{\partial\nu}\frac{1}{r}\,dS+\begin{cases}4\,\pi\phi(P)\\2\,\pi\phi(P)\\0\end{cases},\tag{104.3}$$

according as the point P is inside, on the surface of, or outside the volume of integration. The normal ν is supposed to be directed away from this volume. The only assumption on the function ϕ which is necessary for this result is that it be continuous, with continuous derivatives, throughout the volume of integration. Now let us suppose our origin at a point P of the surface occupied by the double layer, and let us apply the theorem (104.3) to a portion of the volume bounded by the planes $x=\pm a$, $y=\pm b$, $z=\pm h$, the function ϕ being the strength $m(x, y)$ of the double layer, so that ϕ is free from z. The surface occupied by the double layer will divide our parallelepiped into two parts; and we choose, as the volume of integration to which we wish to apply (104.3), the part on the negative side of the double layer. We have, then,

$$\int m\frac{\partial}{\partial\nu}\frac{1}{r}\,dS=\int\left(\operatorname{grad}m\cdot\operatorname{grad}\frac{1}{r}\right)d\tau-\begin{cases}4\,\pi m(Q)\\2\,\pi m(Q)\\0\end{cases},\tag{104.4}$$

according as the point Q of evaluation is on the negative side of, on the surface occupied by, or on the positive side of the double layer. Now each of the volume integrals is merely one of the force components of a volume distribution of matter whose density is proportional to one of the components of **grad** m, and we have seen that these force components are everywhere continuous. If, then, we denote by W_+ and W_- the limiting values approached by $W(Q)$ as the point of evaluation Q approaches the surface occupied by the double layer from the positive and negative sides respectively, we see, from (104.4), that both these limits exist, and that

$$W_+-W_-=4\,\pi m(P),\tag{104.5}$$

where P is the point on the surface which is being approached from the different sides. We see also that the potential W, defined as an improper integral, also exists at the point P, and that

$$W_+-W(P)=2\,\pi m(P),\tag{104.6}$$

so that $W(P)=\frac{1}{2}(W_++W_-)$. These results are expressed in the statement that there is a discontinuity of amount $4\,\pi m(P)$ as we cross the surface at the point P, from the negative to the positive side, and that the value of the potential on the surface itself is the mean of its values on the two sides.

We are now able to determine the behavior of the force components due to a simple-layer distribution of density σ as we cross the surface occupied by the layer. Surrounding a point P on the surface by a small cavity, as usual, we apply Stokes's theorem

$$\oint(\mathbf{A}.\mathbf{ds}) = \int(\mathbf{curl\ A}.\mathbf{dS}),$$

where the vector $\mathbf{A}$, to which we choose to apply this theorem, has components $X = 0$, $Y = \frac{\sigma\cos(\nu\zeta)}{r}$, $Z = -\frac{\sigma\cos(\nu\eta)}{r}$. In applying the theorem, σ and the direction cosines $\cos(\nu\eta)$ and $\cos(\nu\zeta)$ are treated as functions of the parameters (ξ, η), say, on the surface, and r is treated as a function of the three variables (ξ, η, ζ). We readily derive the equation

$$\int\frac{\sigma}{r}[\cos(\nu\zeta)d\eta - \cos(\nu\eta)d\zeta] = \int\frac{F}{r}dS + \int\sigma\frac{\partial}{\partial\xi}\frac{1}{r}dS - \int\sigma\cos(\nu\xi)\frac{\partial}{\partial\nu}\frac{1}{r}dS,$$

where there is no particular interest in writing out the expression F, and we merely observe that it involves the first-order partial derivatives of the density function σ. The second integral on the right-hand side can be written in the form $-\int\sigma\frac{\partial}{\partial x}\frac{1}{r}\,dS$, so that, when multiplied by $-\mu$, it is the integral for the x-component of the force of attraction due to the simple-layer distribution over the cavity. The first and third integrals on the right-hand side are merely potentials due to a volume and double-layer distribution respectively. Our results on the double-layer potential show us that the force components due to a simple-layer potential approach definite limits as we approach the surface on either side, and that

$$X_+ - X_- = -4\,\pi\mu\sigma\cos(\nu x). \qquad (104.7)$$

The normal ν goes from the side of the layer which we denote arbitrarily by $-$ to the other side, which we denote by $+$. It will be remembered that $V = -\mu\int\frac{\sigma\,dS}{r}$ and $X = -\frac{\partial V}{\partial x}$. Furthermore, the force components exist at the point P on the surface itself, and their values there are the means of the limiting values on the two sides. It will be observed that the tangential force components are continuous as we cross the surface, whereas the normal components suffer the discontinuity $-4\,\pi\mu\sigma$ as we cross the surface in the direction of the components. For these results it is necessary only to assume that the surface is smooth at the point in question, that is, that it has a continuously turning normal, and that the density σ has continuous derivatives with respect to the parameters of the surface.

We have seen that the derivatives $\frac{\partial V}{\partial x}$ etc. of the volume potential

$$V = -\mu \int \frac{\rho \, d\tau}{r}$$

exist and that they are continuous everywhere. Now

$$\begin{aligned}\frac{\partial V}{\partial x} &= -\mu \int \rho \frac{\partial}{\partial x} \frac{1}{r} d\tau = \mu \int \rho \frac{\partial}{\partial \xi} \frac{1}{r} d\tau \\ &= \mu \int \frac{\partial}{\partial \xi} \frac{\rho}{r} d\tau - \mu \int \frac{1}{r} \frac{\partial \rho}{\partial \xi} d\tau \\ &= \mu \int \frac{\rho \cos(\nu x)}{r} dS - \mu \int \frac{1}{r} \frac{\partial \rho}{\partial \xi} d\tau,\end{aligned}$$

so that the force component $-\frac{\partial V}{\partial x}$ may be represented as a combination of a simple-layer potential and a volume potential. The density of distribution of the simple-layer potential is $\rho \cos(\nu x)$. If, now, at any internal point P of our volume distribution of matter, we divide the volume into two regions, R_1 and R_2, by a plane passing through P, and denote the potentials of these two regions by V_1 and V_2, we have $V = V_1 + V_2$; and, from our results on the simple-layer and volume potentials, we have

$$\left(\frac{\partial^2 V_1}{\partial x^2}\right)_+ - \left(\frac{\partial^2 V_1}{\partial x^2}\right)_- = -4\,\pi\mu\rho \cos^2(\nu x), \qquad (104.8)$$

where the normal to the dividing plane at P goes from the negative side of the dividing plane to the positive side. It is immaterial which side of the plane we call the positive side. To be definite about it, let us call the side adjacent to the region R_2 the positive side. We have equations similar to (104.8) for the second derivatives with respect to y and z; and, on adding these to (104.8), we have

$$(\Delta_2 V_1)_+ - (\Delta_2 V_1)_- = -4\,\pi\mu\rho.$$

Since, however, any point in the region R_2 is exterior to the region R_1, we have, for all such points, $\Delta_2 V_1 = 0$; so that $(\Delta_2 V_1)_+ = 0$, giving

$$(\Delta_2 V_1)_- = 4\,\pi\mu\rho.$$

Moreover, $(\Delta_2 V_2)_- = 0$, showing that

$$(\Delta_2 V)_- = 4\,\pi\mu\rho.$$

Similarly, $(\Delta_2 V)_+ = 4\,\pi\mu\rho$; and as the force components due to a simple layer have at points on the layer itself the mean of the limits approached on either side, we are led to the result

$$\Delta_2 V = (\Delta_2 V)_- = (\Delta_2 V)_+ = 4\,\pi\mu\rho. \qquad (104.9)$$

At points on the surface of the volume itself,

$$\Delta_2 V = 2\,\pi\mu\rho; \qquad (104.10)$$

and at external points we have $\Delta_2 V = 0$, as stated in (103.4). In proving this result it was necessary to assume that the density ρ was continuous, with continuous partial derivatives $\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z}$.

Although they are not quite so important, we now give briefly the results for the derivatives of the double-layer potential. Using Green's lemma in the form (14.5), we readily see, just as in the derivation of (104.3), that if ψ is any function with continuous first and second partial derivatives with respect to (x, y, z),

$$\int \frac{\Delta_2 \psi}{r}\,d\tau = -\int \psi \frac{\partial}{\partial \nu}\frac{1}{r}\,dS + \int \frac{\partial \psi}{\partial \nu}\frac{1}{r}\,dS - \left\{\begin{matrix} 4\,\pi\psi(P) \\ 2\,\pi\psi(P) \\ 0 \end{matrix}\right\}, \qquad (104.11)$$

according as the point of evaluation P is inside, on the surface of, or outside the volume of integration. Applying this result to the same region (portion of a parallelepiped) used in discussing the discontinuity of the double-layer potential across the surface occupied by the layer, and putting ψ equal to the strength m of the layer, we see that the double-layer potential may be replaced by a volume potential and a simple-layer potential. Using, then, the results already obtained, we find that the derivatives of the double-layer potential $W = \int m \frac{\partial}{\partial \nu}\frac{1}{r}\,dS$ satisfy the equations

$$\left(\frac{\partial W}{\partial x}\right)_+ - \left(\frac{\partial W}{\partial x}\right)_- = 4\,\pi \frac{\partial m}{\partial x} - 4\,\pi \frac{\partial m}{\partial \nu}\cos(\nu x).$$

Hence the normal derivative is continuous,

$$\left(\frac{\partial W}{\partial \nu}\right)_+ = \left(\frac{\partial W}{\partial \nu}\right)_-, \qquad (104.12)$$

whereas the tangential derivative in any direction s is discontinuous, as indicated by the equation

$$\left(\frac{\partial W}{\partial s}\right)_+ - \left(\frac{\partial W}{\partial s}\right)_- = 4\,\pi \frac{\partial m}{\partial s}. \qquad (104.13)$$

We conclude this chapter by giving a résumé of the main results arrived at in the preceding sections.

a. The volume potential $V = -\mu\int\frac{\rho\, d\tau}{r}$. The density ρ being assumed continuous, this exists everywhere and has continuous derivatives. These derivatives are the negative force components. If the partial derivatives of ρ exist and are continuous at an internal point P of the attracting matter, the second partial derivatives $\frac{\partial^2 V}{\partial x^2}$, $\frac{\partial^2 V}{\partial y^2}$, $\frac{\partial^2 V}{\partial z^2}$ exist and are continuous at P, and their sum is $4\pi\mu\rho$. The second partial derivatives with respect to x, y, and z also exist at all points of the surface of the volume, and their sum is $2\pi\mu\rho$.

b. The simple-layer potential $V = -\mu\int\frac{\sigma dS}{r}$. This exists everywhere, and is continuous everywhere on the assumption that the density σ is continuous. The partial derivatives tend to definite limits as we approach a smooth point of the surface from either side, on the assumption that σ has continuous derivatives with respect to the parameters on the surface near this point. The tangential derivatives are continuous as we cross the surface, whereas the normal derivative suffers a discontinuity $4\pi\mu\sigma$ as we cross the surface in the direction of the derivative. At a point on the surface itself the normal derivative is the mean of the two normal derivatives on either side.

c. The double-layer potential $W = \int m\frac{\partial}{\partial\nu}\frac{1}{r}\, dS$. Assuming m to be continuous, this exists everywhere, but is discontinuous as we cross the surface occupied by the double layer. If we assume the strength m to have continuous derivatives with respect to the parameters of the surface near a point P, in the neighborhood of which the surface is smooth, the discontinuity in W as we cross from the negative side of the layer to the positive side is $4\pi m(P)$. The value of W at the point P itself is the mean of the two values on either side of the layer. If we assume, further, that the strength m has continuous second-order derivatives with respect to the parameters of the surface near the point P, the normal derivative of W is continuous as we cross the layer, whereas the tangential derivatives suffer the discontinuity $4\pi\frac{\partial m}{\partial s}$ as we cross from the negative side to the positive side. The values at the point P itself are again the mean of the values on either side.

BIBLIOGRAPHY

The treatment of the various potentials given in the preceding chapter is based on the following excellent paper:

SCHMIDT, E. Bemerkung zur Potentialtheorie. Schwarz-Festschrift, Berlin, 1914.

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KORN, A. Lehrbuch der Potentialtheorie (2 vols.). Berlin, 1899–1901.

The reader is referred also to

LEATHEM, J. G. Volume and Surface Integrals used in Physics. Cambridge Tracts in Mathematics and Mathematical Physics, No. 1.

For the actual evaluation of various potentials the best reference is

ROUTH, E. J. Analytical Statics (Second Edition), Vol. II. Cambridge, 1908.

In this book a complete treatment of the potential of ellipsoids will be found, and there are numerous interesting applications to electrostatics.

EXERCISES

1. State and prove the theorems on the logarithmic potential analogous to those given in the text for the inverse-distance potential. Corresponding to the volume distribution, we now have a surface distribution over a plane area, and the point of evaluation is in the plane of the area. The definition of the logarithmic potential is $V = C - 2\int\sigma\log\frac{1}{r}\,dS$, where σ is the density of distribution over the plane area.

2. Taking the origin O at the center of mass of any volume distribution of matter, the potential at a distant point P is approximately $-\frac{\mu M}{r} - \frac{\mu(A+B+C-3I)}{2r^3}$, where M is the total mass, A, B, C are the principal moments of inertia at the center of mass, I is the moment of inertia about OP, and r is the distance OP.

3. The potential of a uniform ellipsoid whose surface has the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is given, at an internal point (α, β, γ), by the integral $V = -\mu\pi\rho abc\int_0^\infty\left(1-\frac{\alpha^2}{a^2+s}-\frac{\beta^2}{b^2+s}-\frac{\gamma^2}{c^2+s}\right)\frac{ds}{\Delta}$, where $\Delta^2 = (a^2+s)(b^2+s)(c^2+s)$. Verify the fact that this satisfies the equation $\Delta_2 V = 4\pi\mu\rho$.

4. The potential of a uniform ellipsoid at an external point is given by the integral of Exercise 3; but the lower limit, instead of being zero, is λ, where λ is the greatest root of the cubic equation $\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}+\frac{\gamma^2}{c^2+\lambda}=1$. Prove that this satisfies the equation $\Delta_2 V = 0$. Write down the expressions for the force components.

CHAPTER XVII

WAVE MOTION

105. The vibrations of a tightly stretched wire. Let us consider a straight wire of length l which is so tightly stretched that we may neglect the effect of the weight of the wire in comparison with that of its tension when considering the small vibration it executes when it is displaced slightly from its equilibrium position. Choosing, as the x-axis of a system of rectangular coördinates, the equilibrium position of the wire, the origin being at one end, any particle of the wire may be identified by its coördinate x in the equilibrium position. When the wire is displaced from its equilibrium position, the particle x will undergo a vector displacement, (ξ, η, ζ), say, the coördinates of the particle in its displaced position being $(x + \xi, \eta, \zeta)$. Here the components (ξ, η, ζ) of the vector displacement of the particle are functions not only of the variable x, which serves to identify the particular particle of the wire to which we have reference, but of the time variable t. Denoting the element of length of the wire, when in its displaced position, by ds, we have

$$\begin{aligned}(ds)^2 &= (dx + d\xi)^2 + (d\eta)^2 + (d\zeta)^2 \\ &= \left[\left(1 + \frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial x}\right)^2\right](dx)^2,\end{aligned}$$

x playing here the rôle of the independent variable, and t being treated as a constant. If we regard the displacement of the wire to be such that the three derivatives $\left(\frac{\partial \xi}{\partial x}, \frac{\partial \eta}{\partial x}, \frac{\partial \zeta}{\partial x}\right)$ are infinitesimals, we have

$$\frac{ds}{dx} = \left(1 + 2\frac{\partial \xi}{\partial x} + \cdots\right)^{\frac{1}{2}} = 1 + \frac{\partial \xi}{\partial x} + \cdots,$$

or

$$\frac{ds - dx}{dx} = \frac{\partial \xi}{\partial x} + \cdots, \qquad (105.1)$$

where the sign + following an expression indicates that we are neglecting infinitesimals of higher order than those retained in the expression. When the wire is in its displaced position, its tension at the position occupied by the particle identified by x is a vector $\mathbf{T}(x, t)$, tangent to the wire at this point and depending on the values of x and t. The direction of the tangent to the wire at the point occupied by the particle x is given by the direction cosines

$$\left(\frac{dx + d\xi}{ds}, \frac{d\eta}{ds}, \frac{d\zeta}{ds}\right),$$

where t is regarded as constant during the differentiations. The three components of the vector tension are, accordingly,

$$\left(T\frac{1+\dfrac{\partial\xi}{\partial x}}{\dfrac{ds}{dx}}, T\frac{\dfrac{\partial\eta}{\partial x}}{\dfrac{ds}{dx}}, T\frac{\dfrac{\partial\zeta}{\partial x}}{\dfrac{ds}{dx}}\right),$$

where T is the magnitude of the tension, and the partial signs indicate that t is regarded as constant in the differentiations. Since $\left(\frac{ds}{dx}\right)^{-1} = \left(1 + \frac{\partial\xi}{\partial x}\right)^{-1} = 1 - \frac{\partial\xi}{\partial x} + \cdots$, we see that the three components of the tension at the particle x, if we neglect infinitesimals of higher order than the first, are

$$\left(T, T\frac{\partial\eta}{\partial x}, T\frac{\partial\zeta}{\partial x}\right). \tag{105.2}$$

Considering, then, a small element of length of the wire, whose beginning and end particles are identified by the numbers x and $x + dx$ respectively, this element is stretched from a length dx in the equilibrium position to a length ds, given by (105.1), in the displaced position. The change in tension, according to a well-known experimental law known as Hooke's law, is proportional to the relative extension $\frac{ds - dx}{dx}$, the factor of proportionality E being known as the Young's modulus of the wire. We have, therefore,

$$T = P + E\frac{\partial\xi}{\partial x}, \tag{105.3}$$

where P is the magnitude of the tension of the wire in its equilibrium position and is accordingly independent of x. Neglecting any forces acting on the element ds of the wire, when displaced from its equilibrium position, save the tensions at the ends of this element, we have, for the components of the difference of these tensions, the expressions

$$\frac{\partial T}{\partial x}\,dx, \quad \frac{\partial}{\partial x}\left(T\,\frac{\partial \eta}{\partial x}\right)dx, \quad \frac{\partial}{\partial x}\left(T\,\frac{\partial \zeta}{\partial x}\right)dx;$$

and hence, if μ denotes the linear density of the wire, so that the mass of the element dx is $\mu\,dx$, and μ is assumed constant, we have the equations of motion

$$\begin{aligned} \mu\,\frac{\partial^2 \xi}{\partial t^2} &= \frac{\partial T}{\partial x} = E\,\frac{\partial^2 \xi}{\partial x^2}, \\ \mu\,\frac{\partial^2 \eta}{\partial t^2} &= \frac{\partial}{\partial x}\left(T\,\frac{\partial \eta}{\partial x}\right) = P\,\frac{\partial^2 \eta}{\partial x^2}, \\ \mu\,\frac{\partial^2 \zeta}{\partial t^2} &= \frac{\partial}{\partial x}\left(T\,\frac{\partial \zeta}{\partial x}\right) = P\,\frac{\partial^2 \zeta}{\partial x^2}, \end{aligned} \tag{105.4}$$

infinitesimals of higher order than those retained being neglected as before. The first of these three equations governs what are known as the longitudinal vibrations of the wire, the other two governing the transverse vibrations. All three equations are of the type

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2}, \tag{105.5}$$

and we shall proceed to a somewhat detailed discussion of this equation. It is essential to note that the coefficient a^2 is a constant depending on the intrinsic properties of the wire or, in the general case, of the medium through which the disturbance is being propagated. Equation (105.5) defines a particular kind of medium in so far as wave motion in it is concerned.

106. The equation $\dfrac{\partial^2 u}{\partial t^2} = a^2\,\dfrac{\partial^2 u}{\partial x^2}$. Let us first simplify the form of the equation as much as possible by making a linear change of the independent variables (x, t) of the type

$$\xi = lx + mt, \quad \tau = nx + pt.$$

Then

$$\frac{\partial u}{\partial t} = m\,\frac{\partial u}{\partial \xi} + p\,\frac{\partial u}{\partial \tau}, \quad \frac{\partial u}{\partial x} = l\,\frac{\partial u}{\partial \xi} + n\,\frac{\partial u}{\partial \tau},$$

and our equation takes the form

$$m^2\,\frac{\partial^2 u}{\partial \xi^2} + 2\,mp\,\frac{\partial^2 u}{\partial \xi\,\partial \tau} + p^2\,\frac{\partial^2 u}{\partial \tau^2} = a^2 l^2\,\frac{\partial^2 u}{\partial \xi^2} + 2\,a^2 ln\,\frac{\partial^2 u}{\partial \xi\,\partial \tau} + a^2 n^2\,\frac{\partial^2 u}{\partial \tau^2}.$$

If we put $m = \pm\,al$, $p = \mp\,an$, our equation takes the form

$$\frac{\partial^2 u}{\partial \xi\,\partial \tau} = 0, \tag{106.1}$$

provided l and n are both different from zero. They may, without lack of generality, each be put equal to unity, as a change in the values of l and n merely amounts to a change of scale in measuring ξ and τ respectively. The values to be assigned to ξ and τ are, then, given by the equations

$$\xi = x \pm at, \quad \tau = x \mp at. \tag{106.2}$$

It follows, from (106.1), that the variables ξ and τ must be separated from each other in the expression for u in terms of them. In fact, u is the sum of an arbitrary function of ξ and an arbitrary function of τ. Hence the general solution of the equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is

$$u = f_1(x - at) + f_2(x + at), \tag{106.3}$$

where f_1 and f_2 are differentiable functions of their arguments, but are otherwise arbitrary. In order to see the significance of (106.3), let us consider the special case where f_2 is identically zero, a being assumed positive. Then (106.3) becomes

$$u = f_1(x - at). \tag{106.4}$$

This special solution of (105.5) describes what is meant by "wave motion" in the direction of the positive x-axis; for if at the point x_1 at time t_1 the function f_1 has the value h, it will have the same value at the point x_2 at time t_2, provided

$$x_1 - at_1 = x_2 - at_2,$$

that is,

$$\frac{x_2 - x_1}{t_2 - t_1} = a. \tag{106.5}$$

Since x_1 may be any point, and since x_2 assumes continuously, as t_2 varies, all values greater than x_1, the wave form advances continuously with a constant velocity a. We may imagine that we are looking at the wire and that we run our eyes along it with the definite, uniform velocity a in the positive direction of the x-axis. Then the displacement $u = f_1(x - at)$ which we see remains constant. It should be clearly understood that the "wave velocity" a which has just been defined has nothing to do with the actual velocity of any particle of the wire. All that is meant by the term "wave velocity" is that the times and places on the wire where the disturbance expressed by

$u = f_1(x - at)$ repeats itself are connected by the relation (106.5). The fact that u maintains its values without decrease or modification is expressed by the statement that the wave is *undamped.* It advances continuously, keeping a definite form.

Similarly, the special solution

$$u = f_2(x + at) \tag{106.6}$$

represents an undamped wave traveling with uniform velocity a along the x-axis in the negative direction. The complete solution (106.3) indicates a superposition of two such waves.

It is at once obvious that an equation of the form

$$u = f(mx - pt) = f\left[m\left(x - \frac{p}{m}t\right)\right.$$

indicates a wave traveling along the x-axis, the velocity of propagation being $\frac{p}{m}\cdot$ An important special case is

$$u = A \cos (mx - pt + \alpha). \tag{106.7}$$

For x constant this represents a simple harmonic vibration of amplitude A and period $\frac{2\pi}{p}$, the phase being $mx - pt + \alpha$. At any given instant, (106.7) is a simple cosine curve, the distance between two consecutive corresponding points being $\frac{2\pi}{m}\cdot$ This is known as the wave length, and is usually denoted by the symbol λ. The reciprocal of the period gives the number of vibrations per second at any given place, and is known as the frequency, the usual symbol being ν, so that $\nu = \frac{p}{2\pi}\cdot$ We have, then, the following relations:

$$\begin{aligned} &\text{Velocity,} && a = \frac{p}{m}, \\ &\text{Wave length,} && \lambda = \frac{2\pi}{m}, \\ &\text{Frequency,} && \nu = \frac{p}{2\pi}. \end{aligned} \tag{106.8}$$

It follows that the velocity of propagation is equal to the product of the wave length by the frequency.

In the general case the nature of the arbitrary functions f_1 and f_2 of (106.3) has to be determined by two kinds of data.

1. The initial conditions, that is, the values of the displacement u and of the displacement velocity $\frac{\partial u}{\partial t}$ as functions of x when $t = 0$. For example, if the wire is initially displaced and started off from a position of rest, the initial conditions are

$$u = F(x) \quad \text{when} \quad t = 0,$$

and

$$\frac{\partial u}{\partial t} = 0 \quad \text{when} \quad t = 0,$$

for every x. An example is furnished by the vibrations of a plucked violin string. A different set of initial conditions holds when a piano wire is set in vibration by an impulse. Here we have, for every x,

$$u = 0 \qquad \text{when} \quad t = 0,$$

and

$$\frac{\partial u}{\partial t} = G(x) \quad \text{when} \quad t = 0.$$

2. The boundary conditions. For example, the values of u and of $\frac{\partial u}{\partial x}$ may be given for all values of t at the two ends of the wire. As a special case the two ends of the wire may be held fixed, the boundary conditions in this case being

$$u = 0, \quad \frac{\partial u}{\partial x} = 0$$

for all values of t when $x = 0$ and when $x = l$. The problem of determining the functions f_1 and f_2 so as to satisfy the boundary conditions is usually the most difficult part of the question, and we shall consider first an ideal situation where this difficulty does not enter.

107. Wave motion along an unbounded straight line. In this ideal case the wire is supposed to be extended indefinitely in both directions, and we shall suppose the initial conditions to be

$$u = F(x), \quad \frac{\partial u}{\partial t} = G(x), \quad \text{when} \quad t = 0. \tag{107.1}$$

It follows at once, from the general form of the solution

$$u = f_1(x - at) + f_2(x + at),$$

that

$$f_1(x) + f_2(x) = F(x),$$

and

$$-af'_1(x) + af'_2(x) = G(x),$$

where the prime attached to a function indicates differentiation with respect to the argument occurring in it. Integrating the second of these two equations, we find

$$f_1(x) - f_2(x) = -\frac{1}{a}\int^x G(s)ds,$$

the lower limit of the integral being arbitrary. It follows from this and the first of the two previous equations that

$$\begin{aligned} 2f_1(x) &= F(x) - \frac{1}{a}\int^x G(s)ds, \\ 2f_2(x) &= F(x) + \frac{1}{a}\int^x G(s)ds, \end{aligned} \qquad (107.2)$$

so that the solution of the problem is

$$\begin{aligned} u &= f_1(x - at) + f_2(x + at) \\ &= \frac{1}{2}[F(x - at) + F(x + at)] + \frac{1}{2a}\int_{x-at}^{x+at} G(s)\,ds. \end{aligned} \qquad (107.3)$$

It is immediately evident from this expression that, at any instant t, the values of $G(x)$ outside the interval $x - at$ to $x + at$ have no effect upon the displacement u at the point x, and that it is only the values of $F(x)$ at the two end points $x - at$ and $x + at$ of this interval that contribute to the value of u at the point x. It is also evident that if the functions $F(x)$ and $G(x)$ are so related that $F(x) = \pm\frac{1}{a}\int^x G(s)\,ds$, there will be a wave in one direction only along the wire. Thus, if we take the positive sign, we have, from (107.3),

$$\begin{aligned} u &= \tfrac{1}{2}[F(x - at) + F(x + at)] + \tfrac{1}{2}[F(x + at) - F(x - at)] \\ &= F(x + at), \end{aligned}$$

indicating a wave motion in the negative direction along the x-axis.

In particular, if the vibration starts from rest, $G(x)$ is identically zero, and the solution is

$$u = \tfrac{1}{2}[F(x - at) + F(x + at)]. \qquad (107.4)$$

If, now, $F(x)$ is zero outside the interval $-h$ to $+h$ and has the constant value C throughout this interval, we see that two equal waves are propagated in opposite directions, each being a square-cornered "hump" of amplitude $\frac{C}{2}$ and length $2h$. At any point x the displacement becomes zero as soon as the two waves have passed over it.

In the other extreme case, where there is an impulsive start from the undisplaced position, $F(x)$ is identically zero, and

$$\begin{aligned} u &= \frac{1}{2a}\int_{x-at}^{x+at} G(s)\,ds \\ &= \frac{1}{2a}[H(x+at) - H(x-at)]. \quad \left(H(x) = \int^{x} G(s)ds\right) \quad (107.5) \end{aligned}$$

Here we have again the phenomenon of two waves proceeding with constant velocity a in opposite directions along the x-axis; but the amplitudes are opposite, as is seen from the negative sign before the second term.

Let us now suppose again that the initial velocity of all points of the infinite wire is zero, and also that the function $F(x)$ giving the initial displacement is a periodic function, the period being $2l$, say. Then $F(x)$ has a Fourier development

$$F(x) = \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos n\pi\frac{x}{l} + b_n \sin n\pi\frac{x}{l}\right),$$

and the displacement u at any time t is given by

$$\begin{aligned} u &= \tfrac{1}{2}[F(x-at) + F(x+at)] \\ &= \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos n\pi\frac{x}{l} + b_n \sin n\pi\frac{x}{l}\right)\cos n\pi a\frac{t}{l}. \quad (107.6) \end{aligned}$$

The situation is particularly simple when $F(x)$ is an odd function of x, in which case all the coefficients a_n of the Fourier development vanish. The solution is

$$u = \sum_1^\infty b_n \sin n\pi\frac{x}{l}\cos n\pi a\frac{t}{l}. \quad (107.6\,a)$$

This expression of the displacement is said to constitute its analysis into its various harmonics. The nth harmonic is said to have a wave length

$$\lambda_n = \frac{2l}{n},$$

its frequency being

$$\nu_n = \frac{na}{2l}.$$

The velocity of propagation a is the same for all the various harmonics, — a fact which is expressed by the statement that the wave form suffers no distortion. It is well to point out explicitly that (107.6) and (107.6 a), being the results of compounding two wave motions, — one in the positive direction along the x-axis, and one in the negative, — do not themselves represent a wave motion. Each component of (107.6 a) represents the composition of two trains of waves. It is *these waves* which have the wave length λ_n, referred to above.

The vibration given by (107.6 a) was the result of compounding two waves of equal amplitude traveling in opposite directions along the x-axis. It has certain peculiarities which are not possessed by either of these two waves. Thus, let us consider merely the nth harmonic,

$$u = b_n \sin n\pi \frac{x}{l} \cos n\pi a \frac{t}{l}. \tag{107.7}$$

This was the result of compounding the progressive wave

$$u_1 = \frac{b_n}{2} \sin n\pi \frac{x - at}{l}$$

and the regressive wave

$$u_2 = \frac{b_n}{2} \sin n\pi \frac{x + at}{l}.$$

In neither of these waves are there any points x for which u is permanently zero, each particle having a vibration of the same amplitude $\frac{b_n}{2}$, the phase of the vibration changing, however, continuously in the space of a wave length. On the other hand, it is apparent, from (107.7), that the result of compounding the two waves gives a vibration in which the points $x = \frac{ml}{n}$ ($m =$ any positive or negative integer or zero) remain permanently at rest. These points are known as nodes of the vibration, and the vibration is called a stationary or standing wave. The space interval between two consecutive nodes is one half the wave length. At the points midway between consecutive nodes the amplitude of the simple harmonic displacement has its maximum value b_n. Such points are known as loops. The particles x which do not remain permanently at rest all go through their equilibrium and extreme positions at the same times. In a progressive or regressive wave the times at which any particle goes through its equilibrium and extreme positions vary with x.

108. The finite wire. The ends of the wire being $x = 0$ and $x = l$, the functions $F(x)$ and $G(x)$ of (107.1) are given only throughout the range $x = 0$ to $x = l$. So the formula (107.3) has no meaning for values of t carrying $x - at$ or $x + at$ outside that range, unless we extend appropriately the range of values of x for which $F(x)$ and $G(x)$ are defined. The boundary conditions indicate the appropriate definition of these functions outside the original interval $x = 0$ to $x = l$. Assuming that the two end points of the wire are held fixed, we have, from (106.3), for every value of t,

$$\begin{aligned} f_1(-at) + f_2(at) &= 0, \\ f_1(l - at) + f_2(l + at) &= 0. \end{aligned} \qquad (108.1)$$

Eliminating f_2, we obtain

$$f_1(l - at) = -f_2(l + at) = f_1(-l - at).$$

As this relation holds for every value of t, it follows that f_1 is a periodic function of its argument, the period being $2l$. Similarly, f_2 is a periodic function of its argument, with the same period. Since $F(x)$ and $G(x)$ are known over the interval $x = 0$ to $x = l$, $f_1(x)$ and $f_2(x)$ are given over this same interval by the formulas (107.2). Then the first of the relations (108.1) defines f_1 over the range $x = 0$ to $x = -l$, so that $f_1(x)$ is defined for all values of x from $-l$ to $+l$; and as it is periodic, with period $2l$, this suffices to define $f_1(x)$ for all values of x. Similarly, $f_2(x)$ is defined for all values of x. The formulas of the previous section are now applicable, it being observed that $F(x)$ and $G(x)$ are odd functions of x of period $2l$. In fact,

$$\begin{aligned} F(x) &= f_1(x) + f_2(x) \\ &= -f_2(-x) - f_1(-x) \quad \text{[By the first of the relations (108.1)]} \\ &= -F(-x), \end{aligned}$$

and similarly for $G(x)$.

Using the Fourier developments

$$F(x) = \sum_1^\infty b_n \sin n\pi \frac{x}{l},$$

$$G(x) = \sum_1^\infty \beta_n \sin n\pi \frac{x}{l},$$

we have at once, from (107.3), the result

$$u = \sum_1^\infty \sin n\pi \frac{x}{l}\left(b_n \cos n\pi a \frac{t}{l} + \frac{\beta_n l}{an\pi} \sin n\pi a \frac{t}{l}\right). \qquad (108.2)$$

If the vibration starts from rest, we have $\beta_n = 0$, for every n, and we again have the analysis of the vibration of the wire into a series of harmonics. The frequency of the nth harmonic being $\frac{na}{2l}$, we see that the wire can vibrate in pure harmonics only when the frequencies of these simple harmonic vibrations are integral multiples of $\frac{a}{2l}\cdot$ The corresponding wave lengths are integral submultiples of $2l$. The first harmonic is called the fundamental note, and the other harmonics are known as overtones.

The solution given in (108.2) indicates, for any given x, a sum of simple harmonic vibrations. Since u is not now a function of either $x - at$ or $x + at$, (108.2) does not indicate either a progressive or a regressive wave but, rather, a stationary wave. The points x for which x is an integral multiple of $\frac{l}{n}$ are the nodes of the nth harmonic. The quality of the note furnished by the vibrating wire is determined by the relative magnitudes of the amplitudes of the fundamental note and its overtones. Since for the longitudinal waves the velocity of propagation a is equal to $\left(\frac{E}{\mu}\right)^{\frac{1}{2}}$, which is for metals of the order of magnitude 10^5, these vibrations have such a high frequency as to be quite inaudible. It is the transverse vibrations, for which $a = \left(\frac{P}{\mu}\right)^{\frac{1}{2}}$, that furnish the audible notes.

Either of the two waves, progressive and regressive respectively,

$$u_1 = \frac{b_n}{2} \sin n\pi \frac{x - at}{l},$$

$$u_2 = \frac{b_n}{2} \sin n\pi \frac{x + at}{l},$$

which yield, when compounded, the nth harmonic of the note issued by the wire, is usually referred to as a simply periodic train of waves. The essential difference between the train of waves and the stationary, or standing, wave is that in the latter all points have either the same or the opposite phase, whereas in the former the phase varies continuously over a wave length.

109. The musical scale. The musical interval between two pure notes or simple harmonic vibrations is defined as the ratio of the frequency of the note of higher frequency to that of the other. If this fraction, when reduced to its lowest terms, has *small* integers for its numerator and denominator, the combination of the two notes has a pleasant effect upon the ear, and the two notes are said to be consonant or in harmony. The simplest harmony is the octave, for which the musical interval is $2:1$. In the vibrations of a tightly stretched wire with fixed end points, the frequencies of the various harmonics are in the ratio $1:2:3:4 \cdots$, so that the lower-order overtones are in harmony with the fundamental note and with each other. It is for this reason that these overtones are called harmonics. Next to the octave in order of harmony comes the musical fifth, for which the interval is $3:2$. We shall need to consider only those intervals which are less than or equal to 2, the interval $3:1$, for example, being regarded as a combination of the octave $(2:1)$ and the fifth $(3:2)$. Two intervals whose product constitutes an octave are said to be complementary, so that the complement of the fifth, which is known as the fourth, is the interval $4:3$. Next in order of harmony comes the interval $5:3$, which is known as the major sixth, the complementary interval $6:5$ being known as the minor third. Then comes the interval $5:4$, which is known as the major third, its complement, $8:5$, being termed the minor sixth. These are the only intervals which are usually regarded as harmonious. The next interval, $7:4$, is sometimes referred to as the natural seventh, and is closely approximated to by the interval $9:5$, which is a combination of the fifth and the minor third. Starting with low C on the cello, say, the various intervals may be indicated as follows:

← major 6th → ← minor 6th →
1 2 3 4 5 6 7 8
C c g c' e' g' c''
← octave → ← fifth → ← fourth → ← major 3d → ← minor 3d →

The primes refer to the various octaves.

The notes obtainable from any given note by means of consonant intervals form what is known as a *natural scale*. The fewer the number of consonant intervals employed, the simpler is the scale. Thus, using only octaves and fifths, we can, from a fundamental note whose frequency is ν_0, construct the natural scale consisting of the notes whose frequencies are given by the formula

$$\nu = \nu_0 . 2^p . 3^q,$$

p and q being either positive or negative integers or zero. This is known as the Pythagorean scale. It includes, in addition to the octave and fifth, on which it is based, the fourth ($p = 2$, $q = -1$). The major third is not included, but is closely approximated by the interval $\frac{81}{64}(p = -6, q = 4)$, the interval for the major third being $\frac{80}{64}$. The slightly greater interval $\frac{81}{64}$ is known as the Pythagorean third. When limits are placed on the values that the integers (p, q) may assume, we obtain a natural scale consisting of a finite number of notes; but certain consonant intervals are, as we have just seen, missing from the scale.

A simpler scale may be obtained if we alter slightly one of the intervals. In this way is constructed what is known as an artificial, or *tempered*, scale. The tempered scale in general use was advocated by Bach. The octaves are left pure, but it was noticed that twelve fifths are very nearly equal to seven octaves. $\left(\text{The ratio } \frac{3^{12}}{2^{12}.2^{7}} \text{ is, in fact, } 1.014.\right)$ Each fifth is now shortened, so that twelve tempered fifths are exactly equal to seven octaves. (The interval of the tempered fifth is 1.498, instead of the 1.5 of the musical fifth.) The interval $2^{\frac{1}{12}}$ is called a semitone, so that there are twelve semitones in an octave; and the entire piano scale is built up from the semitone. The tempered fourth from *g* to *c′* has the interval $2^{\frac{5}{12}} = 1.335$, which is slightly greater than the natural fourth, 4 : 3. The tempered major third from *c* to *e* has the interval $2^{\frac{1}{3}} = 1.260$, which is slightly greater than the natural major third but less than the Pythagorean third.

110. The composition of wave trains having different frequencies. Let us consider two simply periodic wave trains having the same amplitude and velocity, but with different frequencies, propagated in the same direction along the x-axis. Writing the equations of the two waves in the form

$$u_1 = C\cos\left[2\,\pi\nu_1\left(\frac{x}{a} - t\right) + \alpha_1\right],$$

$$u_2 = C\cos\left[2\,\pi\nu_2\left(\frac{x}{a} - t\right) + \alpha_2\right],$$

where ν_1 and ν_2 are, respectively, the frequencies of the two wave trains, the result of compounding the two wave trains is

$$u = 2\,C\cos\left[\pi(\nu_1 + \nu_2)\left(\frac{x}{a} - t\right) + \frac{\alpha_1 + \alpha_2}{2}\right] \times \cos\left[\pi(\nu_1 - \nu_2)\left(\frac{x}{a} - t\right) + \frac{\alpha_1 - \alpha_2}{2}\right]. \qquad (110.1)$$

This is a complicated wave form advancing without change of velocity a, since u is a function of $x - at$. At any instant t there are two series of points for which $u = 0$. The spacing of one series is $\frac{a}{\nu_1 + \nu_2}$, and that of the other is $\frac{a}{\nu_1 - \nu_2}$ (we are taking $\nu_1 > \nu_2$). If ν_1 is only slightly greater than ν_2, the second factor in (110.1) does not change appreciably for a relatively small interval of time, and we have the phenomenon known as beats, namely, a simply periodic wave train of frequency $\frac{\nu_1 + \nu_2}{2}$ with a simply periodic amplitude of low frequency $\frac{\nu_1 - \nu_2}{2}$.

111. The energy and momentum of a simply periodic wave train. Writing the displacement in the simply periodic wave train in the form

$$u = C \cos 2\,\pi\nu\left(\frac{x}{a} - t\right),$$

the velocity of displacement of the particle x at time t is

$$\frac{\partial u}{\partial t} = 2\,\pi\nu C \sin 2\,\pi\nu\left(\frac{x}{a} - t\right).$$

If ρ is the volume density, and σ the area of cross section of the wire, the kinetic energy of an element of length dx is $2\,\pi^2\rho\sigma\nu^2C^2 \sin^2 2\,\pi\nu\left(\frac{x}{a} - t\right)dx$; and hence the kinetic energy of a length equal to the wave length $\lambda = \frac{a}{\nu}$ is

$$2\,\pi^2\rho\sigma\nu^2C^2\int_0^\lambda \sin^2 2\,\pi\nu\left(\frac{x}{a} - t\right)dx = \pi^2\rho\sigma\nu^2C^2\lambda. \qquad (111.1)$$

If λ is small compared with the unit of length, the kinetic energy per unit of length is, accordingly,

$$\pi^2\rho\sigma C^2\nu^2, \quad \text{or} \quad \frac{\pi^2\rho\sigma C^2a^2}{\lambda^2}. \qquad (111.2)$$

The potential energy of the longitudinal vibrations of the wire is readily obtained from the first of the equations (105.4). The force of restitution on an element dx is $-E\frac{\partial^2 u}{\partial x^2}dx = \frac{4\,\pi^2\nu^2Eu}{a^2}\,dx$, so that the work done against this force is

$$\frac{4\,\pi^2\nu^2E\,dx}{a^2}\int_0^u u\,du = \frac{2\,\pi^2\nu^2Eu^2\,dx}{a^2}. \qquad (111.3)$$

The potential energy of a length of the wire equal to the wave length is, accordingly,

$$\frac{2\,\pi^2\nu^2E}{a^2}\int_0^\lambda u^2\,dx = \frac{\pi^2\nu^2C^2E}{a^2} = \pi^2\rho\sigma C^2\nu^2, \qquad (111.4)$$

since $a^2 = \frac{E}{\mu} = \frac{E}{\rho\sigma}$. On the average, then, the kinetic and potential energies of the longitudinal vibrations are equal. The total energy per unit of length is $2\,\pi^2\rho\sigma C^2\nu^2$. The discussion of the transverse vibrations is the same, the only difference being that P takes the place of E and that $a^2 = \frac{P}{\mu} = \frac{P}{\rho\sigma}$.

Let us now consider the momentum of a simply periodic wave train with velocity of propagation a. Let the waves be incident normally upon a perfectly reflecting wall which is moving with a velocity c in a direction opposite to that of the incident waves. Let us denote the area of the wall by S, and let P be a stationary point in space which is at the instant t at a distance $a + c$ from the wall (see Fig. 36). We shall consider two experimental observations:

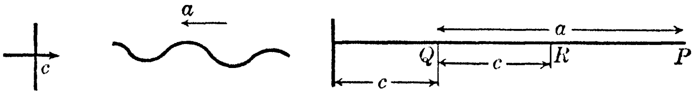

FIG. 36

1. Note the passage of waves past P for two seconds, and let $2\,N$ be the number that pass in this time. At the end of two seconds the wall has reached R, and the reflected waves are back at P. Calling λ the wave length of the incident waves and λ' that of the reflected waves, the space between P and R at the end of the second second contains $\frac{a-c}{\lambda}$ incident waves and $\frac{a-c}{\lambda'}$ reflected ones. Hence

$$\frac{a-c}{\lambda} + \frac{a-c}{\lambda'} = 2\,N = \frac{2\,a}{\lambda},$$

so that

$$\frac{\lambda'}{\lambda} = \frac{a-c}{a+c}. \tag{111.5}$$

2. Consider an instant when the wave front has just reached the wall. One second later, the energy in the space between P and the wall will have reached the latter, which is now at Q. The reflected waves will have reached a distance $a - c$ to the right of Q. Assuming that the amplitude of the waves is not changed by reflection, the energy incident upon the wall in one second is

$$\frac{2\,\pi^2\rho(a+c)SA^2a^2}{\lambda^2}.$$

The energy leaving the wall in the same time is

$$\frac{2\,\pi^2\rho(a-c)SA^2a^2}{\lambda'^2}.$$

The increase in energy of the medium per second as the wall continues to move is therefore

$$2\,\pi^2\rho SA^2a^2\left(\frac{a-c}{\lambda'^2}-\frac{a+c}{\lambda^2}\right)=4\,\pi^2\rho SA^2\,\frac{a^2}{\lambda^2}\left(\frac{a+c}{a-c}\right)c. \quad (111.6)$$

This increase in energy must be due to the fact that work is required to move the wall; that is, that the waves produce a thrust against the wall, even when the wall is stationary. If the magnitude of this thrust is F and the corresponding pressure is p, the work per second required from without in order to make the wall move with velocity c is

$$Fc = pSc = 4\,\pi^2\rho SA^2\,\frac{a^2}{\lambda^2}\left(\frac{a+c}{a-c}\right)c; \quad (111.7)$$

and hence

$$p = 4\,\pi^2\rho A^2\,\frac{a^2}{\lambda^2}\left(\frac{a+c}{a-c}\right). \quad (111.8)$$

If c is small compared with a,

$$p = 4\,\pi^2\rho A^2\,\frac{a^2}{\lambda^2}\left(1+\frac{2\,c}{a}\right). \quad (111.9)$$

If the wall is stationary,

$$p = \frac{4\,\pi^2\rho A^2a^2}{\lambda^2}. \quad (111.10)$$

This is due to the change in momentum of the waves which is due to the wall. Therefore the momentum incident per unit area is

$$\frac{2\,\pi^2\rho A^2a^2}{\lambda^2}, \quad (111.11)$$

in the direction of the wave propagation. So the momentum per unit area per unit length, that is, per unit volume, is

$$\frac{2\,\pi^2\rho A^2a}{\lambda^2} \quad (111.12)$$

in the direction of wave propagation.

If the cylindrical inclosure which contains the medium traversed by the waves has ends of any kind of material, it will not move as a whole; and therefore, since one of the ends may be conceived of as perfectly reflecting, leaving the other of any kind of material, it is clear that the use of a perfect reflector in the discussion does not limit the deductions. When the wall is moving, the energy per unit volume of the medium traversed by both incident and reflected waves is

$$E = 2\,\pi^2 \rho A^2 a^2 \left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) = 2\,\pi^2 \rho A^2 \frac{a^2}{\lambda^2}\left[1 + \left(\frac{a+c}{a-c}\right)^2\right]. \qquad (111.13)$$

If c is small compared with a, this reduces to

$$E = 4\,\pi^2 \rho A^2 \frac{a^2}{\lambda^2}\left(1 + \frac{2\,c}{a}\right), \qquad (111.14)$$

which is numerically equal to the value of p when the same simplifying assumptions are made. If the wall is at rest,

$$E_0 = \frac{4\,\pi^2 \rho A^2 a^2}{\lambda^2}. \qquad (111.15)$$

It follows, therefore, that the kinetic energy per unit volume of the incident (or reflected) train of waves, that is, $\frac{E_0}{4}$, equals $\frac{a}{2}$ times the momentum per unit volume.

In the previous discussion a hypothetical case of plane waves incident normally upon a plane wall — that is, in a cylindrical inclosure closed by plane ends — was described. If the discussion is not so limited, and we consider waves of any undamped type, in all directions, in an inclosure of any shape and material, it may be proved that $p \doteq \frac{E}{3}$.

112. The telegraphic equation. If, in the discussion in § 105, we had assumed, in addition to the factors there considered, a force on each element of length of the vibrating wire whose magnitude was proportional to the velocity of that element, we should have been led to the following generalization of the wave equation (105.5):

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + 2\,b\,\frac{\partial u}{\partial t}. \qquad (112.1)$$

This equation is also met in the problem of the propagation of electric waves along a conducting wire, and it has received the name "telegraphic equation." Forces, such as those now under consideration, whose magnitude is proportional to the velocity and whose direction is opposite to that of the velocity vector are always dissipative. Mathematically this implies that b is always negative. This leads, as will be seen immediately, to the result that the amplitude of the vibration at any point decreases steadily as the time increases. For this reason the telegraphic equation (112.1) is frequently referred to as the equation for damped wave motion along the x-axis. The positive constant $-b$ is then referred to as the damping factor. In order to simplify the discussion of equation (112.1), it is convenient to make a change of the dependent variable by means of a transformation of the type

$$u = e^{mt}U.$$

Upon substitution in (112.1) and division by e^{mt}, we get

$$\frac{\partial^2 U}{\partial t^2} + 2\,m\frac{\partial U}{\partial t} + m^2 U = a^2\frac{\partial^2 U}{\partial x^2} + 2\,bmU + 2\,b\frac{\partial U}{\partial t}.$$

If we set $m = b$, the terms involving the first derivative with respect to t cancel each other. The new form of the equation is

$$\frac{\partial^2 U}{\partial t^2} = a^2\frac{\partial^2 U}{\partial x^2} + b^2 U. \tag{112.2}$$

As in the case of the simpler wave equation, the problem consists of finding a solution of this linear, second-order, partial-differential equation satisfying certain initial and boundary conditions.

Before considering these additional requirements for the solution of the problem, it is convenient to simplify the form of equation (112.2) by introducing the variables

$$\xi = x - at, \quad \tau = x + at \tag{112.3}$$

of § 106. We obtain

$$\frac{\partial^2 U}{\partial\xi\,\partial\tau} + p^2 U = 0. \quad \left(p = \frac{b}{2\,a}\right) \tag{112.4}$$

Let us now suppose the initial conditions to be

$$u = F(x), \quad \frac{\partial u}{\partial t} = G(x), \quad \text{when} \quad t = 0.$$

This implies a knowledge of U and of each of its partial derivatives $\frac{\partial U}{\partial\xi}$ and $\frac{\partial U}{\partial\tau}$ at all points on the straight line $\xi = \tau$. Let us assume for the moment that we know a particular solution of the equation (112.4), and call it V. Then it follows, from (112.4) and the assumption

$$\frac{\partial^2 V}{\partial\xi\,\partial\tau} + p^2 V = 0,$$

that

$$V\frac{\partial^2 U}{\partial\xi\,\partial\tau} - U\frac{\partial^2 V}{\partial\xi\,\partial\tau} = 0.$$

An immediate application of Stokes's theorem then tells us that the line integral

$$\oint\left(U\frac{\partial V}{\partial \xi}\,d\xi + V\frac{\partial U}{\partial \tau}\,d\tau\right),$$

taken around any closed curve in the (ξ, τ) plane, inside which the functions (U, V) and their derivatives are regular, is zero. Taking as our closed curve the straight lines $\xi = \xi_0$, $\tau = \tau_0$ through the fixed point (ξ_0, τ_0), chosen to suit our convenience, and a continuous curve crossing these lines, we have

$$\int_A^B\left(U\frac{\partial V}{\partial \xi}d\xi + V\frac{\partial U}{\partial \tau}d\tau\right) + \int_B^P U\frac{\partial V}{\partial \xi}d\xi + \int_P^A V\frac{\partial U}{\partial \tau}d\tau = 0. \quad (112.5)$$

Here P is the point (ξ_0, τ_0), and A and B are the points where the straight lines through P parallel, respectively, to the axes of coördinates $\xi = 0$ and $\tau = 0$ meet the curve C (see Fig. 37). If we can so determine V that it has the value unity over the two parallels to the axes of coördinates, we find, from (112.5),

$$\int_A^B\left(U\frac{\partial V}{\partial \xi}\,d\xi + V\frac{\partial U}{\partial \tau}\,d\tau\right) + U_A - U_P = 0. \quad (112.6)$$

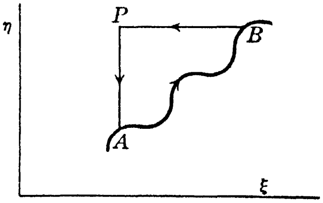

FIG. 37

Equation (112.5) holds for any two solutions U and V of the differential equation (112.4), and hence the relation obtained from (112.5) by interchanging U and V is valid. From this new relation we find

$$\int_A^B\left(V\frac{\partial U}{\partial \xi}\,d\xi + U\frac{\partial V}{\partial \tau}\,d\tau\right) + U_P - U_B = 0. \quad (112.7)$$

From (112.6) and (112.7) we find

$$2\,U_P = U_A + U_B + \int_A^B\left(U\frac{\partial V}{\partial \xi} - V\frac{\partial U}{\partial \xi}\right)d\xi + \int_A^B\left(V\frac{\partial U}{\partial \tau} - U\frac{\partial V}{\partial \tau}\right)d\tau. \quad (112.8)$$

This result expresses the value of U at any point P in terms of its values at the points A and B and its values, together with those of its first partial derivatives, along the curve C, over which the integral in the formula is extended. The method of integration here indicated is known as Riemann's method.

In order that the formula (112.8) may be practical, it is necessary to be able to find a solution V of the differential equation (112.4) which takes the value unity along the straight lines $\xi = \xi_0$, $\tau = \tau_0$. In order to find such a function V, we first introduce the function

$$\zeta = (\xi - \xi_0)(\tau - \tau_0),$$

which vanishes along each of the straight lines in question, and find whether a power series in ζ, whose constant term is unity, will satisfy the differential equation (112.4). Writing

$$V = 1 + \alpha_1\zeta + \alpha_2\zeta^2 + \alpha_3\zeta^3 + \cdots, \tag{112.9}$$

we have $$\frac{\partial V}{\partial \xi} = (\alpha_1 + 2\,\alpha_2\zeta + 3\,\alpha_3\zeta^2 + \cdots)(\tau - \tau_0),$$

$$\begin{aligned}\frac{\partial^2 V}{\partial \xi\,\partial \tau} &= \alpha_1 + 2\,\alpha_2\zeta + 3\,\alpha_3\zeta^2 + \cdots + (2\,\alpha_2 + 3.2\,\alpha_3\zeta + \cdots)(\xi - \xi_0)(\tau - \tau_0)\\ &= \alpha_1 + 2\,\alpha_2\zeta + 3\,\alpha_3\zeta^2 + \cdots + (2\,\alpha_2 + 3.2\,\alpha_3\zeta + \cdots)\zeta. \end{aligned}\tag{112.10}$$

In order that (112.9) may satisfy (112.4) for all values of ζ, the various coefficients of the powers of ζ in the equation obtained by substituting (112.9) and (112.10) in (112.4) must separately vanish. We have, then, the series of equations

$$\alpha_1 + p^2 = 0,\ \ 4\,\alpha_2 + p^2\alpha_1 = 0,\ \ 3^2\alpha_3 + p^2\alpha_2 = 0,\ \ 4^2\alpha_4 + p^2\alpha_3 = 0,\ \text{ etc.}$$

It follows that $\alpha_1 = -\,p^2$, $\alpha_2 = \frac{p^4}{2^2}$, $\alpha_3 = -\frac{p^6}{3^2.2^2}$, $\alpha_4 = \frac{p^8}{4^2.3^2.2^2}$, etc. The desired function V is, accordingly,

$$V = 1 - p^2\zeta + \frac{p^4\zeta^2}{2^2} - \frac{p^6\zeta^3}{2^2.3^2} + \frac{p^8\zeta^4}{2^2.3^2.4^2} - \cdots. \tag{112.11}$$

This series is well known in mathematical physics, and is named the Bessel function J_0 of zero order. In fact, V is $J_0\left(\frac{p\zeta^{\frac{1}{2}}}{2}\right)$.

Let us now avail ourselves of the initial conditions and leave out of consideration the boundary conditions, or, in other words, let us suppose that the medium in which the vibrations are taking place is indefinitely extended in both directions along the x-axis. The initial conditions being

$$u = f(x), \quad \frac{\partial u}{\partial t} = g(x), \quad \text{when} \quad t = 0,$$

we have, from the relation $u = e^{bt}U$, the result that

$$U = f(x), \quad \frac{\partial U}{\partial t} = g(x) - bf(x), \quad \text{when} \quad t = 0.$$

In terms of the variables ξ and τ of (112.3), this says that

$$U = f(\xi), \quad \frac{\partial U}{\partial \tau} = \frac{f'(\xi)}{2} + \frac{g(\xi) - bf(\xi)}{2\,a}$$

along the straight line $\xi = \tau$. Our curve C is now simply the straight line $\xi = \tau$; and so the coördinate ξ of the point A is ξ_0, and that of B is τ_0. Along the straight line of integration in the formula (112.8) $d\xi = d\tau$, and so we have from that formula

$$\begin{aligned} 2U(\xi_0, \tau_0) &= U(\xi_0) + U(\tau_0) + \int_{\xi_0}^{\tau_0} \left[U\left(\frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \tau}\right) - V\left(\frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \tau}\right)\right] d\xi \\ &= f(\xi_0) + f(\tau_0) + \frac{1}{a}\int_{\xi_0}^{\tau_0} \left(V\frac{\partial U}{\partial t} - U\frac{\partial V}{\partial t}\right) d\xi. \end{aligned}$$

We may now drop the zero subscript and revert to the original independent variables (x, t). We have

$$\begin{aligned} 2\,U(x, t) = f(x - at) + f(x + at) &+ \frac{1}{a}\int_{x-at}^{x+at} V[g(\xi) - bf(\xi)]d\xi \\ &- \frac{1}{a}\int_{x-at}^{x+at} \left[f(\xi)\,\frac{\partial V}{\partial t}\right] d\xi. \end{aligned}$$

In the integrals on the right, V is given as a function of ζ which has, along the straight line $\xi = \tau$, the value $(\xi - \xi_0)(\xi - \tau_0)$. Hence the term $\frac{\partial V}{\partial t}$ in the second integral has the value

$$\begin{aligned} \frac{\partial V}{\partial t} &= a\left(\frac{\partial V}{\partial \tau} - \frac{\partial V}{\partial \xi}\right) \\ &= a\,\frac{dV}{d\zeta}\,(\xi - \xi_0 - \tau + \tau_0) \\ &= a(\tau_0 - \xi_0)\,\frac{dV}{d\zeta} \end{aligned}$$

along the line $\xi = \tau$, and the solution of the problem is

$$\begin{aligned} 2\,U(x,t) &= f(x - at) + f(x + at) + \frac{1}{a}\int_{x-at}^{x+at} V(\zeta)[g(\xi) - bf(\xi)]d\xi \\ &\quad - 2\,at\int_{x-at}^{x+at} f(\xi)\,\frac{dV}{d\zeta}\,d\xi, \quad [\zeta = (\xi - \xi_0)(\xi - \tau_0)] \qquad (112.12) \end{aligned}$$

where V is given by the formula (112.11).

It is convenient to remark at this point that the method of integration outlined above for the telegraphic equation (112.1) is applicable without any modification to the simpler wave equation (105.5). The auxiliary function V is in this case simply the constant unity, p being zero; and we have the result (remembering that $b = 0$)

$$2\,U(x, t) = f(x - at) + f(x + at) + \frac{1}{a}\int_{x-at}^{x+at} g(\xi)d\xi,$$

which was obtained by a different method in (107.3). The advantage of the more general method outlined above is that we are not limited by it to "initial conditions" implying a knowledge of the unknown function and its time derivative when $t = 0$. All we need to know are the values of the unknown and one of its partial derivatives along any curve C crossing the two parallels to the (ξ, τ) axes that pass through the point (ξ_0, τ_0) at which the value of the unknown function u is desired. For example, we may deal with a situation where an impulse of known magnitude (this magnitude being a function of x) is applied at known instants to the various points x. Such a situation arises, for example, when a railroad train passes over a bridge. If the motion of the train is given by $x = K(t)$, the equation of the curve C in the (ξ, τ) plane is

$$\xi + \tau = 2\,K\left(\frac{\tau - \xi}{2\,a}\right),$$

and the "initial conditions" are

$$u = 0, \quad \frac{\partial u}{\partial t} = g(x),$$

along this curve.

In order to get an insight into the physical meaning of the formula (112.12), let us assume that the initial disturbance is confined to a finite portion of the wire, — say, from $-h$ to $+h$. Then it is apparent from the formula (112.12) that, at any given instant t_1, U is zero for all points x such that either $x - at_1 > h$ or $x + at_1 < -h$. Both ends of the wave travel, therefore, with constant velocity a in opposite directions. This is the same phenomenon that takes place in the case of the simple wave equation. The situation is, on the other hand, quite different for points not at the ends of the waves. In the case of the simple wave equation a point x came to rest immediately after both waves had passed over it; but this is not the case here. In fact, let us consider an instant $t_1 > \frac{h}{a}$ and a point x_1 between $h - at_1$ and $-h + at_1$. Then, since $x_1 + at_1 > h$ and $x_1 - at_1 < -h$, the value of U at (x_1, t_1) is given by the two integrals in (112.12). If, for simplicity, we assume the initial velocity to be zero, U will be given by the formula

$$2\,U = -\frac{b}{a}\int_{x-at}^{x+at} V(\zeta) f(\xi)\,d\xi - 2\,at\int_{x-at}^{x+at} f(\xi)\,\frac{dV}{d\zeta}\,d\xi.$$

The displacement does not, therefore, become zero after the waves have passed over any point. In other words, neither the progressive nor the regressive wave has a definite end, although they have definite beginning points.

It is apparent, from the formula (112.12), that if f and g are odd functions of their arguments, U will be zero for $x = 0$ and all values of t. This is the case in the reflection of an electromagnetic wave, for example, by a metal surface.

It is evident that a simple solution of (112.2) is

$$U = C \sin (mx - nt),$$

where the relation

$$n^2 = a^2m^2 - b^2 \tag{112.13}$$

holds. This implies that the damping factor depends on the frequency, and involves the phenomenon known as distortion, where waves of different frequencies are unequally damped so that the character or form of the wave varies as it advances. In order to see clearly how the phenomena of dispersion and distortion occur, let us consider a solution of (112.1) of the type $u = Ue^{-ipt}$, U being a function of x alone. Then we must have

$$\frac{d^2U}{dx^2} = -\left(\frac{p^2 - 2\,bip}{a^2}\right)U. \tag{112.14}$$

Writing

$$\frac{p^2 - 2\,bip}{a^2} = (\alpha + i\beta)^2, \quad (\beta > 0)$$

we have

$$U = Ce^{i(\alpha + i\beta)x} = Ce^{-\beta x}e^{i\alpha x}.$$

(Only one of the two solutions of (112.14) is allowable if u is to remain finite as $x \rightarrow \infty$.)

Then

$$u = Ce^{-\beta x}e^{i(\alpha x - pt)}. \tag{112.15}$$

Here $\alpha^2 - \beta^2 = \dfrac{p^2}{a^2}$, $\alpha\beta = -\dfrac{bp}{a^2}$, so that both α and β are functions of p, that is, of the frequency. The fact that α is a function of the frequency implies dispersion. The fact that the damping factor β is a function of the frequency implies distortion.

113. Group velocity and dispersion. Let us now consider the composition of two simply periodic wave trains of the same amplitude, but possessing different frequencies and velocities of propagation. This composition is of importance in connection with media for which the wave velocity is a function of the frequency. The differential equations for the wave trains being

$$\frac{\partial^2 u}{\partial t^2} = a_1{}^2 \frac{\partial^2 u}{\partial x^2}$$

and

$$\frac{\partial^2 u}{\partial t^2} = a_2{}^2 \frac{\partial^2 u}{\partial x^2},$$

we may indicate the two wave trains by the equations

$$u_1 = C \cos\left[2\,\pi\nu_1\left(\frac{x}{a_1} - t\right) + \alpha_1\right],$$

$$u_2 = C \cos\left[2\,\pi\nu_2\left(\frac{x}{a_2} - t\right) + \alpha_2\right].$$

The result of compounding these is

$$u = u_1 + u_2 = 2\,C \cos\left[\pi x\left(\frac{\nu_1}{a_1} + \frac{\nu_2}{a_2}\right) - \pi t(\nu_1 + \nu_2) + \frac{\alpha_1 + \alpha_2}{2}\right]$$

$$\times \cos\left[\pi x\left(\frac{\nu_1}{a_1} - \frac{\nu_2}{a_2}\right) - \pi t(\nu_1 - \nu_2) + \frac{\alpha_1 - \alpha_2}{2}\right]. \quad (113.1)$$

The frequency of the simply periodic wave train indicated by the first factor is

$$\nu = \frac{\nu_1 + \nu_2}{2}, \quad (113.2)$$

and the velocity of propagation is

$$a = \frac{a_1 a_2(\nu_1 + \nu_2)}{\nu_1 a_2 + \nu_2 a_1}. \quad (113.3)$$

The other factor indicates a simply periodic wave train with frequency

$$\nu' = \frac{\nu_1 - \nu_2}{2} \quad (113.4)$$

and the velocity of propagation

$$a' = \frac{a_1 a_2(\nu_1 - \nu_2)}{\nu_1 a_2 - \nu_2 a_1}. \quad (113.5)$$

If ν_1 is very nearly equal to ν_2, the period of this wave train is very long, and (113.1) indicates a simply periodic wave train with slowly varying amplitude. This amplitude is simply periodic, its period being $\frac{1}{\nu'}$; and its maximum value at any point occurs at a time $\frac{x}{a'}$ later than it occurs at $x = 0$. The essential thing to notice is that the wave form changes as it advances, but that the maximum amplitude reappears at intervals.

If, now, we consider a wave train composed of simply periodic wave trains with frequencies between ν and $\nu + \Delta\nu$, the velocity a being a function of the frequency ν, we have what is known as the phenomenon of dispersion. The velocity a' with which the maximum amplitude is propagated is known as the *group velocity* of the wave train. Writing, in the formula (113.5), $a_2 = a$, $a_1 = a + \Delta a$, $\nu_2 = \nu$, $\nu_1 = \nu + \Delta\nu$, and letting $\Delta\nu$ tend to zero, we have, for the group velocity of a wave train of frequency ν, the result

$$a' = \frac{a}{1 - \dfrac{\nu}{a}\dfrac{da}{d\nu}} \qquad (113.6)$$

$$= a\left(1 + \frac{\nu}{a}\frac{da}{d\nu}\right) \qquad (113.7)$$

if the rate of change of velocity of propagation with respect to frequency is small.

The accompanying figure will illustrate the concept of group velocity. Two simply periodic wave trains with different wave lengths $\lambda_1 = \frac{a_1}{\nu_1}$, $\lambda_2 = \frac{a_2}{\nu_2}$ are shown with their crests in coincidence at a point A at a given instant. This coincidence of crests will occur again at a later time, when the faster train has gained a distance $\lambda_1 - \lambda_2$ over the slower, that is, after a lapse of time $\frac{\lambda_1 - \lambda_2}{a_1 - a_2}$. In this interval of time the crest B of the slower wave train will have advanced a distance $\frac{a_2(\lambda_1 - \lambda_2)}{a_1 - a_2}$, so that the condition (of crest coincidence) which was at A at time $t = 0$ reappears at a distance beyond A equal to

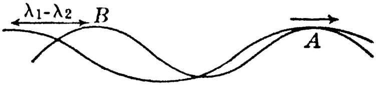

FIG. 38

$$\frac{a_2(\lambda_1 - \lambda_2)}{a_1 - a_2} - \lambda_2 = \frac{\lambda_1 a_2 - \lambda_2 a_1}{a_1 - a_2}$$

after the lapse of time $\frac{\lambda_1 - \lambda_2}{a_1 - a_2}$. The quotient of this distance by the elapsed time, that is, $\frac{\lambda_1 a_2 - \lambda_2 a_1}{\lambda_1 - \lambda_2}$, is the group velocity of the two wave trains. Putting $\lambda_1 = \frac{a_1}{\nu_1}$, $\lambda_2 = \frac{a_2}{\nu_2}$, we find again the formula (113.5). It should be observed that the "group," as such, does not advance. It exists at a certain place, fades away, reappears reversed in a new position, again fades away, then reappears in its original form at another place, and so on.

J. J. Thomson's model to illustrate dispersion. This consists of a row of beads, each of mass m, on a cord whose tension is T and at a distance l apart. Suspended from the beads are a series of bobs, each of mass M, having the same free period. Let y_n be the horizontal displacement of the nth bead, and z_n that of the nth bob. Denoting by μ the force required to displace a bob a unit distance sidewise from its equilibrium position, we have the equations

$$m\frac{d^2y_n}{dt^2} = T\left[\frac{(y_{n+1} - y_n)}{l} - \frac{(y_n - y_{n-1})}{l}\right] + \mu(z_n - y_n),$$

$$M\frac{d^2z_n}{dt^2} = \mu(y_n - z_n).$$

However, y_n being a function of x, we may put

$$y_{n-1} = y_n - l\frac{dy_n}{dx} + \frac{1}{2}l^2\frac{d^2y_n}{dx^2} - \cdots,$$

$$y_{n+1} = y_n + l\frac{dy_n}{dx} + \frac{1}{2}l^2\frac{d^2y_n}{dx^2} + \cdots;$$

so that, on the assumption that l is an infinitesimal, the two equations of motion simplify to

$$m\frac{d^2y_n}{dt^2} = Tl\frac{d^2y_n}{dx^2} + \mu(z_n - y_n),$$

$$M\frac{d^2z_n}{dt^2} = \mu(y_n - z_n).$$

Writing $m = \rho l$, $M = \rho' l$, $\mu = \mu' l$, and using the sign of partial differentiation, since y_n is a function of the two independent variables (x, t), we have

$$\rho\frac{\partial^2y_n}{\partial t^2} = T\frac{\partial^2y_n}{\partial x^2} + \mu'(z_n - y_n),$$

$$\rho'\frac{\partial^2z_n}{\partial t^2} = \mu'(y_n - z_n).$$

If waves $y = C\cos(rx - st - \alpha)$ pass along the row of beads, $\frac{\partial^2y_n}{\partial t^2} = -s^2y_n$, $\frac{\partial^2y_n}{\partial x^2} = -r^2y_n$. There are similar equations for z_n. Equating the two values obtained for the ratio $\frac{y_n}{z_n}$ from our two equations, we have

$$r^2 = \frac{s^2(\rho\rho's^2 - \mu'\rho - \mu'\rho')}{T(\rho's^2 - \mu')}.$$

The velocity v of propagation of the waves is $v = \frac{s}{r}\cdot$ The velocity of waves in the cord, if there were no suspended bobs, would be $v_0 = \left(\frac{T}{\rho}\right)^{\frac{1}{2}}$.

Hence
$$\frac{v_0^2}{v^2}=\frac{s^2-\dfrac{\mu'(\rho+\rho')}{\rho\rho'}}{s^2-\dfrac{\mu'}{\rho'}}.$$

But the period of vibration of a bob, if allowed to swing freely, is $2\pi\left(\frac{\rho'}{\mu'}\right)^{\frac{1}{2}}$. Call this τ, and let $\frac{\mu'}{\rho'}=n^2$, so that $n^2=\frac{4\pi^2}{\tau^2}$. Then
$$\frac{v_0^2}{v^2}=\frac{s^2-\dfrac{n^2(\rho+\rho')}{\rho}}{s^2-n^2}=1-\frac{n^2\dfrac{\rho'}{\rho}}{s^2-n^2}.$$

The suspended bobs may be regarded as corresponding to particles of matter suspended in the ether. Then $\frac{v_0^2}{v^2}$ is the square of the index of refraction.

114. The general wave equation. The equation $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$, which was discussed in § 106, is a particular instance of the more general equation
$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \qquad (114.1)$$
which is known as the wave equation, and which dominates the propagation of waves, such as sound waves, electromagnetic waves, etc. Before discussing equation (114.1) in any detail, it will be convenient to make some general remarks on the linear partial-differential equation of the second order in which there are n independent variables and one dependent variable. This may be written
$$\sum_1^n a_{rs}\frac{\partial^2 u}{\partial x_r\,\partial x_s}+F\left(x,\,u,\,\frac{\partial u}{\partial x}\right)=0. \qquad (114.2)$$

In our applications the coefficients a_{rs} will be constants; but it will be seen that the discussion is valid if they involve the independent variables x, provided they do not involve the dependent variable u. Adopting the notation
$$p_r=\frac{\partial u}{\partial x_r},\quad p_{rs}=\frac{\partial^2 u}{\partial x_r\,\partial x_s},\quad (r,\,s=1,\,2,\,\cdots,\,n) \qquad (114.3)$$
our equation takes the form
$$\Sigma a_{rs}p_{rs}+F(x,\,u,\,p)=0. \qquad (114.4)$$

The problem is to find a function u of the n independent variables x satisfying the equation such that u and *one* of its partial derivatives (p_1, say) take assigned values over a given spread
$$\phi(x_1,\,x_2,\,\cdots,\,x_n)=0$$

of $n-1$ dimensions in the space of n dimensions constituted by the n independent variables x. It is at once apparent that a knowledge of u and one of its partial derivatives at all points of the spread $\phi=0$ implies a knowledge of the other partial derivatives of the first order. For the points x on the spread $\phi=0$ are expressible as functions of $n-1$ independent parameters, $(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$, say; and the identical relation

$$du = p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n,$$

where u and the x's are given functions of the parameters α, is equivalent to $n-1$ distinct equations expressing the p's in terms of the α's. If one of the p's is given in terms of the α's, these relations enable us to solve for the others.

To illustrate these general remarks, consider the case, already treated, of the vibrations of an indefinitely extended wire. Here $n=2$, $x_1=x$, $x_2=t$, $a_{11}=a^2$, $a_{12}=0$, $a_{22}=-1$, $F=0$. The spread $\phi=0$ is, in the simplest case, t or $x_2=0$, and the single parameter α_1 on this spread may be taken as x or x_1. We are given the values of u and of $\frac{\partial u}{\partial t}$ when $t=0$, and this implies a knowledge of the values of $\frac{\partial u}{\partial x}$ when $t=0$. In fact, since x and t are independent, t may be put equal to zero before differentiation with respect to x is performed; and so $\frac{\partial u}{\partial x}$ is found by merely differentiating, with respect to x, the given function of x which yields the values of u when $t=0$. Of course, this example is trivial, but it may serve to illustrate the general idea.

The values of all the first partial derivatives $(p_1, p_2, \cdots, p_n)$ being known as functions of the $n-1$ independent parameters $(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ on the spread $\phi=0$, we may use the identical relations

$$dp_r = \sum_s \frac{\partial p_r}{\partial x_s} dx_s = \sum_s p_{rs} dx_s, \quad (r=1, 2, \cdots, n) \quad (114.5)$$

to calculate the values of the second-order partial derivatives p_{rs} over this spread. We have, in fact, from (114.5), for each value of r the $n-1$ equations

$$\frac{\partial p_r}{\partial \alpha_q} = \sum_s p_{rs} \frac{\partial x_s}{\partial \alpha_q}. \quad (q=1, 2, \cdots, n-1)$$

Writing these in the form

$$\sum_{s=1}^{n-1} p_{rs} \frac{\partial x_s}{\partial \alpha_q} = \frac{\partial p_r}{\partial \alpha_q} - p_{rn} \frac{\partial x_n}{\partial \alpha_q}, \quad (q=1, 2, \cdots, n-1) \quad (114.6)$$

we can solve for $(p_{r1}, p_{r2}, \cdots, p_{r\,n-1})$ as functions of the parameters α and p_{rn}. Denoting by J_r, for the moment, $(-1)^{r-1}$ times the determinant formed by erasing the rth column of the matrix of $n-1$ rows and n columns whose element in the lth row and mth column is $\frac{\partial x_m}{\partial \alpha_l}$, the solution of (114.6) has the form

$$p_{rs} = \frac{J_s p_{rn}}{J_n} + \text{a known function of the } \alpha\text{'s}.$$

$$(r = 1, 2, \cdots, n; \;\; s = 1, 2, \cdots, n-1) \quad (114.7)$$

Putting $r = n$ in these equations, we have

$$p_{ns} = \frac{J_s p_{nn}}{J_n} + \text{a known function of the } \alpha\text{'s},$$

and as $p_{ns} = p_{sn}$, substitution of this in (114.7) gives

$$p_{rs} = \frac{J_s J_r}{J_n^2}\, p_{nn} + \text{a known function of the } \alpha\text{'s}. \quad (114.8)$$

Hence all the second-order derivatives are expressible, on the spread $\phi = 0$, in terms of one of them (p_{nn}, say) and the parameters α.

We have tacitly assumed that J_n does not vanish identically. If it did, the argument would apply by merely making one of the other coördinates x play the rôle assigned above to x_n, unless, indeed, all the Jacobian determinants J_r vanished identically on the spread $\phi = 0$. This contingency cannot, however, arise, since then the spread would have less than $n-1$ dimensions, the functions x of the parameters α not being distinct.

If we substitute the expressions which we have given in (114.8) for the second-order derivatives p_{rs} in the differential equation (114.4), we have an equation of the type

$$A p_{nn} + B = 0, \quad (114.9)$$

which serves to determine p_{nn}, A being given by the equation

$$A = \sum_{r,s} a_{rs} J_r J_s. \quad (114.10)$$

Now the $n-1$ equations

$$\sum_{s=1}^{n} \frac{\partial \phi}{\partial x_s} \frac{\partial x_s}{\partial \alpha_q} = 0, \quad (q = 1, 2, \cdots, n-1)$$

obtained by differentiating the equation $\phi = 0$ of the spread with respect to the parameters α of the points on it, tell us that

$$\frac{\partial \phi}{\partial x_1} : \frac{\partial \phi}{\partial x_2} : \cdots : \frac{\partial \phi}{\partial x_n} = J_1 : J_2 : \cdots : J_n.$$

Hence p_{nn}, and from it all the second-order **derivatives** p_{rs}, may be determined on the spread $\phi = 0$, *unless* $A = 0$, or, equivalently,

$$\sum_{r,s}^{n}{}_{1} a_{rs} \frac{\partial \phi}{\partial x_r} \frac{\partial \phi}{\partial x_s} = 0. \tag{114.11}$$

The spreads $\phi = 0$ satisfying this first-order, second-degree, partial-differential equation are known as the characteristic spreads of the second-order partial-differential equation under discussion. The third-order partial derivatives may be treated in the same way; and it develops that, provided only that $\phi = 0$ is *not* a characteristic spread of the differential equation, the third-order partial derivatives, and, in succession, the partial derivatives of any order, may be calculated on the spread $\phi = 0$. An application of Taylor's theorem giving the expansion, in a power series, of a function of several variables then tells us that there is a unique solution of the problem and, in fact, indicates a method of approximating the value of the unknown function u at any point $(x_1, x_2, \cdots, x_n)$.

115. The physical significance of the characteristic spreads. When the spread $\phi = 0$, over which the values of u and its first-order partial derivatives are known, is a characteristic spread, the problem set is either impossible or indeterminate. To be possible, it is necessary that the B which occurs in equation (114.9) be zero. If this is true, the values of p_{nn} over the characteristic spread may be assigned arbitrarily. The physical significance of the characteristic spreads may be clearly seen if we consider again, for a moment, the case of the vibrations of an indefinitely extended wire. There is the obvious, if trivial, solution of the differential equation, namely, $u = 0$, corresponding to a state of rest of the wire. If, now, the wire is set in vibration by striking it at a point, the disturbance will travel in both directions along the wire. A point x, remote from the center of disturbance, will remain at rest until the wave reaches it at a time $t = \psi(x)$, say. Using a plane on which we plot the variables (x, t), the curve

$$\phi(x, t) \equiv t - \psi(x) = 0 \tag{115.1}$$

divides the plane into two regions. In the region below the curve we have $t < \psi(x)$, and the displacement u of the point x is zero for these times, since the wave has not yet reached the point. In the region above the curve, on the other hand, $t > \psi(x)$, and u is, in general, different from zero, at least for points near the curve (115.1). Assuming that u varies continuously, with continuous partial derivatives of the first order, from the undisturbed to the disturbed state, we have

$$u = 0, \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} = 0$$

on the curve $\phi(x, t) = 0$. Corresponding to these "initial conditions" there exists, in addition to the actual solution giving the wave, the trivial solution $u = 0$. Hence the curve (115.1) which represents the *wave front* is a characteristic curve of the wave equation; for the problem of determining u does not have a unique solution.

The same argument applies without modification to any wave propagation. Thus, for the general wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

the wave front is a moving surface which is a characteristic spread of the differential equation. The wave front satisfies, therefore, the first-order partial-differential equation

$$\left(\frac{\partial \phi}{\partial t}\right)^2 = a^2\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]. \tag{115.2}$$

If $\phi = 0$ is solved for t in the form $\phi \equiv t - \psi(x, y, z) = 0$, ψ satisfies the equation

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \frac{1}{a^2}. \tag{115.3}$$

A particular solution of (115.3), having the property of being symmetrical about the origin, is found by first introducing space polar coördinates, when (115.3) takes the form

$$\left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \psi}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2 \theta}\left(\frac{\partial \psi}{\partial \phi}\right)^2 = \frac{1}{a^2}.$$

If ψ is a function of r alone, this gives $\frac{d\psi}{dr} = \pm\frac{1}{a}$, so that $\psi = \pm\frac{r}{a} + \text{constant}$. The additive constant is of no significance;

and we see, then, that a solution of (115.3) which has the property of being symmetrical about the point (α, β, γ) is

$$\psi = \frac{[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]^{\frac{1}{2}}}{a}, \tag{115.4}$$

the corresponding moving wave front being the spherical surface

$$\phi \equiv t - \psi = t - \frac{r}{a} = 0, \tag{115.5}$$

where r is the distance from the point (α, β, γ) which is the center of disturbance. The integral (115.5) of (115.2) involves three arbitrary constants (α, β, γ) and is a *complete* integral of the equation. The general integral corresponding to a wave front $F(x, y, z) = 0$ at time $t = 0$ is found by getting the envelope of the spherical wave fronts (115.5) as the point (α, β, γ) traces out the surface $F(x, y, z) = 0$. It is, in fact, evident that as t tends to zero this envelope tends to the surface $F(x, y, z) = 0$. Only that portion of the envelope which is on the side of $F = 0$ toward which the wave is being propagated is to be considered. This method of obtaining the successive positions of the wave front from its position at any instant ($t = 0$, say) is known as *Huygens's principle.*

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This text has some interesting diagrams illustrating the subject.

EXERCISES

1. Discuss the small vibrations of a mass m at the end of an elastic wire, treating separately the two cases (1) horizontal vibrations, the effect of gravity being neglected, (2) vertical vibrations.

HINT. The displacement of any particle of the wire being denoted by ξ, $\frac{\partial^2\xi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2\xi}{\partial x^2}$; and a solution of this is $\xi = A \sin qx \cos pt$, provided $q = p\left(\frac{\rho}{E}\right)^{\frac{1}{2}}$. The tension in the wire is $E\sigma\frac{\partial\xi}{\partial x}$. At the end $x = L$ of the wire we have to equate the tension to the product of the mass of the attached particle and its acceleration $\frac{\partial^2\xi}{\partial t^2}$; and we derive $\tan qL = \frac{E\sigma q}{Mp^2}$, an equation serving to determine p. Successive approximations may be arrived at by first setting $\tan qL = qL$ (which implies that qL is small) and then, for a better approximation, setting $\tan qL = qL\left(1 + \frac{q^2L^2}{3}\right)$. A graphical solution of the equation may also be employed.

2. One extremity of a stretched wire is fixed in space, and the other is agitated so as to have the obligatory motion $\xi = A \sin qt$. Determine the forced vibration of the wire.

3. Show that when a tightly stretched wire is plucked aside at one of its nodes, those harmonics which also have a node at that point are not sounded.

4. The ends of a stretched wire of length l are fastened to two equal masses M controlled by springs of strength μ allowing transversal vibrations. If the string is plucked at its middle point, the period p of vibration will be given by $ma \tan \frac{\pi l}{pa} = \frac{p\mu}{2\pi} - \frac{2\pi M}{p}$, where m is the linear density and ma^2 the tension of the wire.

CHAPTER XVIII

THE LORENTZ–EINSTEIN TRANSFORMATION; TENSORS

116. Measurement of fundamental quantities. Many fundamental questions concerning the concepts involved in mechanics — namely, space, time, and matter — have been purposely ignored in the previous chapters. Attention must now be called especially to them.

One group of questions arises in connection with the concepts of the distance between two points, the length of a material rod, the duration of an interval of time, the mass of a moving body, etc. The first point to emphasize is that by the words "length," either of a space interval or of a time interval, and "mass" there is meant necessarily a measured quantity. So we must describe how the measurements are made, that is, how numbers are assigned to lengths of space and time and to mass.

Another group of questions arises in connection with the nature of the geometry of the space in which we live. This space appears to us to be three-dimensional, but the question arises as to whether it is Euclidean or not. The only way to decide this question is to make deductions from the axioms of Euclidean geometry concerning the metric properties of plane and solid figures, and then to make measurements of actual figures. If there is agreement between the results of our measurements and the theorems deducible from the axioms of Euclidean geometry, it will be convenient to assume that our space is Euclidean; if there is disagreement, this assumption is not legitimate. In the latter case it is possible that our measurements may give a clue as to the type of geometry we should develop in order to describe satisfactorily the phenomena we observe. All the questions, therefore, refer ultimately to measurements. Methods of measurement must be postulated.

In the previous chapters the frame of reference selected was a Euclidean inertial one, that is, one for which the law of inertia holds. Another way of expressing this condition is to say that an observer in such a space would not be conscious of any constraints. This would obviously exclude a gravitational field and a frame of reference attached to a body rotating with respect to an inertial frame. By definition, all observers in this space are at rest in it, and we shall make our postulates with reference to it.

We shall postulate, first, that if one point is visible from another, a ray of light may be traced from one to the other. The trace of this ray we shall call the straight line joining the two points. We shall further postulate that there is a definite property of a specified monochromatic beam of light, called its wave length, and that this can be compared with the length of a straight line. Therefore this wave length is the primary standard of length.

To secure secondary standards, we postulate the existence of rigid bodies. One of these may be in the form of a bar or rod, and its length may be compared with the standard wave length. This length may be subdivided etc. Then, to measure the length of any straight line, the method of superposition is postulated, it being definitely understood that there is no relative motion between the material body on which the straight line is described, the standard bar, and the observer. If there is such relative motion, the method cannot be applied.

It is necessary to note that if the orientation of the beam of light in the frame of reference, or that of the object on which the straight line is described, is changed, we are, strictly speaking, dealing with a new problem. We postulate that the same number would be obtained for the length of the line as before. In other words, there is no change in the length of an object when it is oriented differently, — the space is isotropic. (This does not say anything at all about possible changes in the object, but emphasizes that by "length" we always mean the result of a measurement.)

The fundamental problem concerning time is to devise a method for giving a number to an interval of time. We have recourse to our intuitive concepts. Certain motions appeal to us as *periodic*; for example, the motion of a pendulum, of the balance wheel of a watch, of a weight suspended by a spiral spring, etc. We could choose arbitrarily one of these motions and define the word "periodic" by reference to it. Again, certain rectilinear motions appeal to us as *uniform*; and we could choose one of these and define the word "uniform" by reference to it. Either of these definitions would lead to a method for giving numbers to intervals of time. It is interesting to consider each process in some detail.

1. Let us define "periodic" by reference to the vibrations of a particular pendulum in vacuo. By means of it a clock may be driven, and thus time intervals may be measured. Other clocks may be made (driven by pendulums, springs, etc.) which have the same rate as the standard clock, and, in addition, the hands may all be set alike. These clocks may then be distributed among the various observers in any one system.

One use of such clocks might be to measure velocities of moving objects, or of sound, of light, etc. An interesting question would be whether the velocity of light is the same in different directions, for different distances, when the source of light is in the system and outside, etc. Actual experiments indicate that this velocity in vacuo is always the same, that is, that all observations give the same number if the same units are used. In the centimeter-second system this number is 3×10^{10}. For purposes of comparing (or standardizing) two clocks which are at a distance apart, we are led to make the *postulate* that if observers in any one system measure the velocity of light in a vacuum, regardless of the location or character of the source, the same number is obtained. With this postulate, two observers at a distance apart can synchronize their clocks, even if they have not been compared previously (as described above). Let the observer at the point A, say, have the standard clock, and let an observer at any point B have any clock; let the former send a light signal to the latter, the signal being supposed to be immediately reflected at B so as to return to A. Let the observer at A note on his clock the readings t_A and t'_A, representing, respectively, the time when the signal is sent and the time when it returns. Let the observer at B note the reading t_B on his clock when the signal is received and reflected. Then, if the latter alters the rate of his clock and the setting of its hands (if necessary), so that whenever the test is made,

$$t_B - t_A = t'_A - t_B,$$

that is,

$$2\,t_B = t'_A + t_A, \tag{116.1}$$

the two clocks are synchronous.

2. Let us define "uniform velocity" by reference to the velocity of light in a vacuum, in a specified direction, from a specified source stationary in the system. Then equal intervals of time are those required for this light to traverse equal lengths in this direction, and a clock may be rated accordingly. The time scale would depend upon the number arbitrarily assigned the velocity of light. Observers could investigate the velocity of light from different sources, in different directions, etc.; and, finally, they would be led to make the same postulate as before concerning the velocity of light. Then two observers could synchronize their clocks by the same method as that described above.

117. Measurement of space and time intervals in a moving system. If there is relative motion between the observer and the object whose length is to be measured, or on which phenomena in time are occurring, new problems arise. For example, certain phenomena occur in a moving railway car and are noted by observers on the car and on the ground. What conclusions can be drawn as to the connection between the two sets of observations? If a man walks along the car with a velocity v' as noted by observers on the car, and if the car has a velocity v as noted by observers on the ground, is the velocity of the man, as noted by these latter observers, $v + v'$? This particular question involves the following problem: If observers in the car lay off a measured distance L, what number for this would observers on the ground obtain when the car is moving with velocity v? Nothing has been said up to the present about a method for measuring the length of a moving body. An obvious method is to have a number of observers along the path of the moving object, and for two of these to mark on the ground *simultaneously* two points which coincide with the two ends of the object as it passes, and then, later, to measure the distance between these two marks. To simplify the problem, we make the *postulate* that a system K' having a *uniform* velocity with reference to an inertial system is itself an inertial system. (Thus observers in K' may introduce the same system of measurement, both of distance and time, as we have described above.) We assume that observers in K' have obtained their standard bars and clocks from observers in K, so that they are using the same units and are using the same number for the velocity of light.

We postulate, further, that if observers in K measure the length of a certain body, and if this is transferred to K', observers in the latter would, by their measurements, obtain the same number for its length. Or if observers in K measure the length of a certain body in K', and if the body is transferred to K and its length measured by observers in K', the same numbers are obtained, provided that the orientation of the length measured with reference to the line of relative velocity is the same in both cases. In more general language, our postulate is that the only fact concerning two systems which have a uniform relative velocity, that observers in the two systems can ascertain, is that there is such a uniform relative velocity. Nothing can be learned about the absolute motion of either one — the words have no meaning; and both sets of observers would deduce the same "laws of nature."

118. The Lorentz-Einstein transformation. If some "event" is seen and noted by both sets of observers, it will be given certain coördinates, (x, y, z, t), say, by those in K, and (x', y', z', t') by those in K'. It is important to deduce the mathematical connection between the two sets of numbers:

$$\begin{aligned} x' &= f_1(x, y, z, t), \qquad y' = f_2(x, y, z, t), \quad \text{etc.,} \\ x &= F_1(x', y', z', t'), \quad y = F_2(x', y', z', t'), \quad \text{etc.} \end{aligned} \tag{118.1}$$

In order to do this let us choose as the axis of X (and X') a line parallel to the line of relative motion and common to both systems, and let us take the axes of Y and Z (and of Y' and Z') perpendicular to this line and to each other. The axes Y and Y' (and Z and Z') are parallel, meaning that if observers in K draw a series of planes perpendicular to X, and if observers in K' draw a series of planes perpendicular to X', both series of planes will appear to both sets of observers to be parallel to each other. The origins may be chosen anywhere in the common line X and X'. When, in the course of the motion, the two origins coincide, let all the clocks in both systems be put at "zero time." At the same instant let a light signal be flashed in the common direction of OX and $O'X'$. Its progress may be noted by observers in both systems. When the signal reaches an observer in K at the point A, at a time t as observed by him, it will be perceived by the observer in K' at the point A' (coincident with A in the space K). Let us denote his clock reading by t'. If V is the number adopted for the velocity of light (it being the same in both systems), we have

$$a = Vt, \quad a' = Vt'.$$

Here a and a' are the coördinates in the direction of motion, in the systems K and K' respectively, of the point A (or A'). If, now, we assume that the coördinates perpendicular to the direction of motion have no effect upon the connection between (x, t) and (x', t'), the problem is a two-dimensional one, and the most general assumption we can make about the connection between the space-time coördinates of an event in the two systems is

$$x' = f(x, t, v), \quad y' = y, \quad z' = z, \quad t' = g(x, t, v). \tag{118.2}$$

We have indicated explicitly that the magnitude of the relative velocity may be expected to enter the equations of connection. It is usual to assume immediately that the functions f and g are necessarily linear functions of the coördinates (x, t), but this is not necessary.*

Thus, from the relations (118.2), we have

$$\frac{dx'}{dt'} = \frac{\dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial t}}{\dfrac{\partial g}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial g}{\partial t}}. \tag{118.3}$$

Assuming, as above, that to the velocity $\frac{dx}{dt} = V$ of light corresponds the velocity $\frac{dx'}{dt'} = V$, we find, on using (118.3), that

$$V^2 \frac{\partial g}{\partial x} + V \frac{\partial g}{\partial t} = V \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}. \tag{118.4}$$

The assumption that the systems have a relative velocity v means that when $\frac{dx}{dt} = 0$, $\frac{dx'}{dt'} = -v$, and that when $\frac{dx'}{dt'} = 0$, $\frac{dx}{dt} = v$. Using these postulates in (118.3), we find

$$\frac{\partial f}{\partial t} = -v \frac{\partial g}{\partial t} \tag{118.5}$$

and

$$\frac{\partial f}{\partial t} = -v \frac{\partial f}{\partial x}. \tag{118.6}$$

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* See Alonzo Church's article entitled "Uniqueness of the Lorentz Transformation," *American Mathematical Monthly* (1924), Vol. XXXI, pp. 376–382.

On integrating (118.5) with respect to t, we find that $f + vg$ is a function of x alone. Using (118.5) and (118.6) in (118.4), we find

$$V^2 \frac{\partial g}{\partial x} = - v \frac{\partial f}{\partial x}, \tag{118.7}$$

which yields, on integration with respect to x, the result that

$$f + \frac{V^2}{v} g$$

is a function of t alone. Hence f and g are each the sum of a function of x and a function of t. In other words, the variables x and t are separable in each of the functions f and g. Writing f in the form $X_1 + T_1$, and g in the form $X_2 + T_2$, and denoting by attached primes the derivative of a function with respect to its argument, we have, from (118.5), $T'_1 + vT'_2 = 0$; whence

$$T_1 + vT_2 = a, \tag{118.8}$$

where a is a constant. From (118.6) we have $T'_1 = - vX'_1$; and, as different variables enter the two sides of this equation, each of these sides must be a constant. This tells us that T_1 and X_1 are linear functions of t and x respectively. It follows immediately, from (118.7) and (118.8), that T_2 and X_2 are linear functions of t and x respectively. We have so chosen our origins of measurement of (x, t) and (x', t') that when $x = 0$ and $t = 0$, then $x' = 0$ and $t' = 0$, and so there are no constant terms in the linear functions giving x' and t' in terms of x and t. In fact, if we write

$$X_1 = \beta x + \alpha,$$

then

$$T_1 = - v\beta t + \delta,$$

$$T_2 = \beta t + \epsilon,$$

$$X_2 = \frac{-v}{V^2} \beta x + \eta,$$

where $(\alpha, \delta, \epsilon, \eta)$ are constants of integration.

From these results we have

$$x' = f(x, t, v) = X_1 + T_1 = \beta(x - vt) + (\alpha + \delta),$$

$$t' = g(x, t, v) = X_2 + T_2 = \beta\left(t - \frac{vx}{V^2}\right) + (\epsilon + \eta).$$

The choice of our origins of measurement enables us to set $\alpha + \delta = 0$, $\epsilon + \eta = 0$, and it only remains to determine the value of the constant β. This is done by means of the postulate that there is no way of distinguishing between the two systems. In other words, any equations derived by observers in K would apply also to observations made by observers in K' if v is replaced by $-v$. Thus the equations

$$x' = \beta(x - vt), \quad t' = \beta\left(t - \frac{vx}{V^2}\right) \tag{118.9}$$

must imply the equations

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{vx'}{V^2}\right), \tag{118.10}$$

where γ is the same function of $-v$ that β is of v. If we formally solve equations (118.9), we get exactly equations (118.10) with

$$\gamma^{-1} = \beta\left(1 - \frac{v^2}{V^2}\right). \tag{118.11}$$

If we assume that there is no way of distinguishing between the two directions on the x-axis, it follows at once that β must be an even function of v, that is, that $\gamma = \beta$. For equations (118.9) must still hold when we change the signs of x, x', and v, the value of β being unaffected. Putting $\gamma = \beta$ in (118.11), we find

$$\beta = \left(1 - \frac{v^2}{V^2}\right)^{-\frac{1}{2}}. \tag{118.12}$$

Equations (118.9) and (118.12), giving the connection between the space-time coördinates of an event in the two systems of reference K and K', are fundamental and are known as the Lorentz-Einstein formulas of transformation. The physical significance of the determination of the value of β which we have given in (118.12) will be clearer from the following argument. Starting from equations (118.9), which are a consequence of the fact that there is no way of distinguishing between the two systems and that the velocity of light is the same in both, let us imagine a rod stationary in K, along the axis of X, one of its ends being at the origin and the other falling at $x = L$, so that the length of the rod as measured in the system K is L. To find its length as observed in K', two observers in K' must note the values of x' corresponding to its two ends simultaneously, that is, at the same instant $t' = \tau'$, say. For the end $x = 0$ we have $x' = -v\tau'$, from (118.10); and for the end $x = L$ we have $x' = \beta\left(1 - \frac{v^2}{V^2}\right)L - v\tau'$, from (118.10) and (118.11). Hence the length of the rod, as measured in K', is

$$L\beta\left(1 - \frac{v^2}{V^2}\right). \tag{118.13}$$

If the same rod is placed in K' along the axis of X', one end being at the origin, the other will fall at $x' = L$, since we have made the fundamental postulate that the two systems have identical properties. To find the length of the rod as measured in the system K, two observers must note the positions of its two ends simultaneously, that is, at the same instant $t = \tau$, say. For the end $x' = 0$ we have $x = v\tau$, from (118.9), and for the end $x' = L$ we have $x = v\tau + \frac{L}{\beta}$; so that the length as measured in K is $\frac{L}{\beta}$. The postulate that there is no way of distinguishing between the two systems forces us to take

$$L\beta\left(1 - \frac{v^2}{V^2}\right) = \frac{L}{\beta},$$

giving again the value for β which appears in (118.12).

119. Immediate consequences of the Lorentz-Einstein transformation. There are several immediate and important consequences of the formulas

$$x' = \beta(x - vt), \quad y' = y, \quad z' = z, \quad t' = \beta\left(t - \frac{vx}{V^2}\right), \qquad \beta = \left(1 - \frac{v^2}{V^2}\right)^{-\frac{1}{2}} \tag{119.1}$$

to which attention must now be called.

(1) $$x'^2 - V^2t'^2 = x^2 - V^2t^2,$$

or, more generally,

$$x'^2 + y'^2 + z'^2 - V^2t'^2 = x^2 + y^2 + z^2 - V^2t^2. \tag{119.2}$$

This says that the velocity of light in each system has the same numeric value V. It is therefore closely connected with one of the fundamental postulates from which the formulas (119.1) were derived. The postulate itself tells us that when

$$x^2 + y^2 + z^2 - V^2t^2 = 0, \quad x'^2 + y'^2 + z'^2 - V^2t'^2 = 0.$$

It does not of itself furnish the relation (119.2).

(2) If $x = 0$ then $x' = -\beta vt$, and $t' = \beta t$; whence $\frac{x'}{t'} = -v$, which merely expresses the fundamental postulate that if the velocity of the system K' relative to K is v, the velocity of the system K relative to K' is $-v$. In each case the velocity of the moving system is measured by observers in the other system.

(3) The rod referred to in § 118 has the length L when measured by observers in the system with respect to which it is

stationary; whereas if its length is measured by observers in the other system, the number obtained is $\frac{L}{\beta}\cdot$ So long as $v < V$, β is greater than unity. If $v = V$, β is infinite. If it were possible for v to be greater than V, β would be imaginary. This means simply that, by the system of measurement specified, it is impossible to have a motion so rapid that v ever comes out greater than V (V has the properties of an infinite number). In general language, then, a body in motion with reference to an observer will appear to him to be contracted along the line of motion. Lengths perpendicular to the line of motion will, on the other hand, appear to be unaltered.

(4) Let there be a man in K' who is doing something requiring a certain interval of time; for example, smoking a cigarette. Let him begin at time t' and finish at time $t' + \tau'$. A man in K who is a cigarette-smoker may be supposed to associate this same time interval τ' with smoking a cigarette. But let the observers in K note on their clocks the time taken by the man in K' to finish his cigarette. Using the equation

$$t = \beta\left(t' + \frac{vx'}{V^2}\right),$$

we have, since x' is constant,

$$\Delta t = \beta \Delta t' = \beta\tau'. \tag{119.3}$$

That is, since $\beta > 1$, the observers in K would say that the man in K' took a longer time to smoke his cigarette than that to which they were accustomed.

120. Measurement of the mass of a moving particle. We can now discuss the questions raised concerning relative motion, or at least some of them.

Let there be a particle in motion whose velocity, as measured by an observer in K', has the components (w'_x, w'_y, w'_z). What are the components (w_x, w_y, w_z) of the velocity as noted by an observer in K? We find at once

$$\frac{dx}{dt} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{V^2}\frac{dx'}{dt'}},$$

so that

$$w_x = \frac{w'_x + v}{1 + \frac{vw'_x}{V^2}}. \tag{120.1}$$

Similarly, $$w_y = \frac{w'_y}{\beta\left(1 + \frac{vw'_x}{V^2}\right)}, \quad w_z = \frac{w'_z}{\beta\left(1 + \frac{vw'_x}{V^2}\right)}. \qquad (120.2)$$

The important fact to notice in connection with these formulas is that it is only when v and w'_x are infinitesimal in comparison with V that the denominators in the expressions given above can be put equal to unity. In this case, however,

$$w_x = w'_x + v, \quad w_y = w'_y, \quad w_z = w'_z.$$

These are the classical formulas for the composition of simultaneous velocities.

It has been proved that the numbers assigned to lengths, intervals of time, and velocities depend upon the relative motion between the object measured and the observer. The question at once arises, Does the value obtained for the mass of a body depend upon the relative motion between it and the observer? The experiment by which mass is defined is one involving the interaction (impact) of two spheres moving along the same straight line. It is postulated that the ratio $\frac{u_2 - u'_2}{u'_1 - u_1}$ is a constant for the two bodies, where (u_1, u_2) and (u'_1, u'_2) are the velocities of the two bodies before and after impact respectively. This ratio is written

$$\frac{u_2 - u'_2}{u'_1 - u_1} = \frac{m_1}{m_2}, \qquad (120.3)$$

so that $$m_1u_1 + m_2u_2 = m_1u'_1 + m_2u'_2. \qquad (120.4)$$

Then, if a number m_1 is assigned arbitrarily to one body, the corresponding number m_2 is defined by (120.4) for the other. This mode of defining the method of determining the mass of a body is equivalent to saying that we assign an arbitrary number to the sum $m_1 + m_2$ and specify that $m_1u_1 + m_2u_2$ remains unchanged during interaction. That is, we write down the two equations

$$m_1 + m_2 = C, \quad m_1u_1 + m_2u_2 = \text{constant}, \qquad (120.5)$$

in which C is a specified arbitrary constant. Now, u_1 and u_2 are to be observed before and after interaction (impact); and since (120.4) holds, the ratio $\frac{m_1}{m_2}$ is calculable. The first of the equations (120.5) determines m_1 and m_2 separately.

If this experiment is observed by observers in K and K', the former would determine the masses by the equations (120.5), and the latter by the similar equations

$$m'_1 + m'_2 = C', \quad m'_1u'_1 + m'_2u'_2 = \text{constant}. \tag{120.6}$$

The primes are now reserved to distinguish the two systems, and do not refer to the situation after impact. If there is no relative motion between the two systems, — that is, if in the transformation equations $v = 0$, — then $C' = C$. Let the line of interaction (impact) be parallel to the x-axis, so that

$$u' = \frac{u - v}{1 - \dfrac{uv}{V^2}}. \tag{120.7}$$

(This equation is derived from (120.1) by interchanging the rôle of the two systems K and K'.) It is convenient to obtain a symmetrical expression for $\frac{u-v}{u'}$. Let dt_0 be the measure for an interval of time occurring on a body having the velocity u as measured in K (and u' as measured in K'), dt_0 being determined by observers on the body. Then it follows, from (119.3), that if dt is the measure of the same interval of time as determined by observers in K,

$$dt = \alpha\, dt_0, \quad \alpha = \left(1 - \frac{u^2}{V^2}\right)^{-\frac{1}{2}}. \tag{120.8}$$

Similarly,

$$dt' = \alpha'\, dt_0, \quad \alpha' = \left(1 - \frac{u'^2}{V^2}\right)^{-\frac{1}{2}}. \tag{120.9}$$

Hence $\frac{dt'}{\alpha'} = \frac{dt}{\alpha}$. It follows, from the equations (119.1), that

$$dt' = \beta\, dt\left(1 - \frac{vu}{V^2}\right) = \beta\, dt\,\frac{u - v}{u'}. \quad \text{[By (120.7)]}$$

Hence

$$\frac{u - v}{u'} = \frac{\alpha'}{\beta\alpha}. \tag{120.10}$$

Returning now to the equations for m_1 and m_2, we have, since βv is determined by the relative velocity of the two systems K and K' and is independent of the velocities of the two interacting masses,

$$(m_1 + m_2)\beta v = \beta v C = \text{constant}.$$

Since β is independent of the velocities of the two masses, $(m_1u_1 + m_2u_2)\beta$ is unchanged by the impact. Hence, on subtraction, $\beta m_1(u_1 - v) + \beta m_2(u_2 - v)$ remains unchanged by the interaction. Therefore, on substituting from (120.10), we have $\frac{m_1\alpha'_1u'_1}{\alpha_1}+\frac{m_2\alpha'_2u'_2}{\alpha_2}$ unchanged by the interaction. But $m'_1u'_1+m'_2u'_2$ remains, by definition, unchanged. Hence, since u'_1 and u'_2 are arbitrary, we must have

$$m'_1=\frac{m_1\alpha'_1}{\alpha_1},\quad m'_2=\frac{m_2\alpha'_2}{\alpha_2},$$

or, in general,

$$m'=\frac{m\alpha'}{\alpha}. \tag{120.11}$$

If the particle were at rest in K', u' would be equal to 0 and α' would be equal to 1. Denoting the corresponding value of m' by m_0, and calling it the rest mass of the particle, we have

$$m=\alpha m_0.$$

Here, however, $u=v$, since $u'=0$ (see (120.7)), so that $\alpha=\beta$; and we have, finally,

$$m=\beta m_0. \tag{120.12}$$

That is, the measure of the mass by the observers in K is β times as great as the measure by the observers in the system K' in which the particle is at rest. As $\beta=\left(1-\frac{v^2}{V^2}\right)^{-\frac{1}{2}}$, we see that, as v approaches V in value, m becomes very large. This means that the force required to increase the velocity becomes very large when the velocity is near that of light, and becomes infinite when the velocity of light is reached. In other words, the velocity of light cannot be exceeded, — a result which was previously obtained from other considerations.

If the ratio $\frac{v}{V}$ is small, $m=m_0$, giving the classical result that the mass of a particle is independent of its velocity. Carrying the approximation a stage further, we have

$$m=\beta m_0=\left(1-\frac{v^2}{V^2}\right)^{-\frac{1}{2}}m_0=m_0+\frac{1}{2}\frac{m_0v^2}{V^2}. \tag{120.13}$$

The "additional mass" is therefore $\frac{T}{V^2}$, where T is the kinetic energy of the particle as noted by the observers in K.

It should be observed that no matter how the velocity of a particle relative to a system K is changing, its rest mass at any instant may be defined by the introduction of a system K'

which has at that instant the same velocity as the particle relative to the system K, — there being, however, this difference, that the velocity of the particle relative to K is changing, whereas the velocity of K' relative to K is, of course, constant. In other words, in order that the α' of (120.11) may be unity, it is sufficient that u' be zero at the instant in question without being permanently zero. We shall make the postulate that the rest mass of a particle is always the same. Thus, when a particle changes its velocity relative to a system K, so that a new system K' must be introduced in order to measure the rest mass, we postulate that there is no change in m_0. In other words, the rest mass of a particle is a property of the particle, and has nothing to do with the system in which it is measured.

The velocity of a particle of mass m being denoted by $\mathbf{w}$ (and the magnitude of the velocity being w), the linear momentum $\mathbf{G}$ is defined by the equation

$$\mathbf{G} = m\mathbf{w} = \alpha m_0 \mathbf{w}, \tag{120.14}$$

where m_0 is the rest mass of the particle and $\alpha = \left(1 - \frac{w^2}{V^2}\right)^{-\frac{1}{2}}$. The force on the particle, measured by observers in the system K, will therefore be

$$\mathbf{F} = \frac{d\mathbf{G}}{dt} = m_0 \frac{d}{dt}(\alpha \mathbf{w}). \tag{120.15}$$

Two special cases may be considered:

CASE I. *The force is in the direction of motion.* The time rate of change of the vector $\alpha\mathbf{w}$ is, accordingly, directed along $\mathbf{w}$, and this implies that $\mathbf{w}$ has a constant direction. Taking $\mathbf{w}$ along the x-axis, we have

$$w_y = 0, \quad w_z = 0, \quad \alpha = \left(1 - \frac{w_x^2}{V^2}\right)^{-\frac{1}{2}},$$

and

$$F_x = m_0 \frac{d}{dt}(\alpha w_x) = m_0 \alpha^3 \frac{dw_x}{dt} = m\alpha^2 \frac{dw_x}{dt}. \tag{120.16}$$

The ratio of the force to the acceleration is accordingly $m\alpha^2$.

CASE II. *The force is perpendicular to the direction of motion.* The scalar product of $\frac{d}{dt}(\alpha\mathbf{w})$ by $\mathbf{w}$ is here zero, so that

$$w^2 \frac{d\alpha}{dt} + \alpha(\dot{\mathbf{w}}.\mathbf{w}) = 0,$$

yielding, on integration,

$$\alpha^2 w^2 = \text{constant};$$

so that the speed w is constant, or, what is the same thing, α is constant. Then

$$\mathbf{F} = m_0\alpha\,\frac{d\mathbf{w}}{dt} = m\,\frac{d\mathbf{w}}{dt}, \qquad (120.17)$$

and the ratio of the force to the acceleration is m, as in the classical theory.

In general, the activity or power of the force — that is, the rate at which it does work on the particle — is

$$(\mathbf{F}.\mathbf{w}) = m_0 w^2\,\frac{d\alpha}{dt} + \frac{1}{2}\,m_0\alpha\,\frac{dw^2}{dt}. \qquad \text{[By (120.15)]}$$

Upon substituting for α its value $\left(1 - \frac{w^2}{V^2}\right)^{-\frac{1}{2}}$, we find, for the activity $(\mathbf{F}.\mathbf{w})$, the expression

$$m_0 V^2\,\frac{d\alpha}{dt}. \qquad (120.18)$$

Defining the kinetic energy T of the particle by the equation

$$(\mathbf{F}.\mathbf{w}) = \frac{dT}{dt}, \qquad (120.19)$$

we have
$$T = m_0 V^2\alpha + C,$$

where C is a constant of integration. When the particle is at rest, $\alpha = 1$; and, denoting by T_0 the corresponding kinetic energy, we have

$$T_0 = m_0 V^2 + C,$$

whence
$$T - T_0 = m_0 V^2\alpha - m_0 V^2. \qquad (120.20)$$

If the kinetic energy is measured from the condition of rest, — that is, if T_0 is put equal to zero, — we have

$$T = m_0 V^2\alpha - m_0 V^2 = mV^2 - m_0 V^2. \qquad (120.21)$$

It follows immediately that

$$m = m_0 + \frac{T}{V^2}, \qquad (120.22)$$

showing that mass and energy are of the same nature.

121. Transformation of Maxwell's equations for the electromagnetic field. We leave to the reader the proof of the fact that the equations of Maxwell for the electromagnetic field in empty space, namely,

$$\operatorname{curl} \mathbf{H} = \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \operatorname{div} \mathbf{H} = 0,$$

$$\operatorname{curl} \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div} \mathbf{E} = 0,$$

where $V^2 = \frac{c^2}{\epsilon\mu}$, transform into

$$\operatorname{curl} \mathbf{H}' = \frac{\epsilon}{c}\frac{\partial \mathbf{E}'}{\partial t}, \qquad \operatorname{div} \mathbf{H}' = 0,$$

$$\operatorname{curl} \mathbf{E}' = -\frac{\mu}{c}\frac{\partial \mathbf{H}'}{\partial t}, \qquad \operatorname{div} \mathbf{E}' = 0,$$

where

$$\begin{aligned} &H'_x = H_x, \quad H'_y = \beta\left(H_y + \frac{\epsilon v E_z}{c}\right), \quad H'_z = \beta\left(H_z - \frac{\epsilon v E_y}{c}\right), \\ &E'_x = E_x, \quad E'_y = \beta\left(E_y - \frac{\mu v H_z}{c}\right), \quad E'_z = \beta\left(E_z + \frac{\mu v H_y}{c}\right). \end{aligned} \qquad (121.1)$$

If v is small compared with V, so that we can neglect squares of $\frac{v}{V}$, we have $\beta = 1$, and the equations (121.1) simplify to

$$\mathbf{H}' = \mathbf{H} - \frac{\epsilon}{c}\,[\mathbf{v}.\mathbf{E}], \quad \mathbf{E}' = \mathbf{E} + \frac{\mu}{c}\,[\mathbf{v}.\mathbf{H}]. \qquad (121.2)$$

If Maxwell's equations are regarded as the fundamental equations of electromagnetism, the equations (121.1) tell us that when observers in K find the electric and magnetic intensity vectors to be **E** and **H** respectively, observers in K' will measure these as **E**$'$ and **H**$'$ respectively. Thus, when observers in K find that there is no magnetic intensity, observers in K' will not agree with them in general. If we postulate that the laws of nature must be the same for the two systems, this tells us that the statement "The electric intensity is zero" cannot be the expression of a physical fact. The statement that both the electric and magnetic intensities are zero is, however, a possible statement; for if they are zero in the system K, the equations (121.1) tell us that they will be zero in any system K' possessing a uniform velocity relative to K.

EXERCISES

1. Show that the equations of Lorentz for a stationary system K in which a charge $\rho\, d(x, y, z)$ has the velocity w, namely,

$$\operatorname{curl} \mathbf{H} = \frac{1}{V}\left(\frac{\partial \mathbf{E}}{\partial t} + \rho \mathbf{w}\right), \quad \operatorname{div} \mathbf{H} = 0,$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{V}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div} \mathbf{E} = \rho$$

(using units defined by $\epsilon = 4\pi$, $\mu = 4\pi$, $c = 4\pi V$), transform into the same equations in the primed letters **E**′ and **H**′, whose values in terms of **E** and **H** are given in (121.1). There is the additional equation of definition

$$\rho' = \beta\rho\left(1 - \frac{vw_x}{V^2}\right), \tag{121.3}$$

and ϵ, μ, and c are supposed to have the values given above.

2. Show that $\rho' w'_x = \beta\rho(w_x - v)$, $\rho' w'_y = \rho w_y$, $\rho' w'_z = \rho w_z$.

3. Show that

$$\begin{aligned} &d(x', y', z') = \beta d(x, y, z) - \beta v d(y, z, t), \quad d(x', y', t') = d(x, y, t), \\ &d(y', z', t') = \beta d(y, z, t) - \frac{\beta v}{V^2} d(x, y, z), \quad d(x', z', t') = d(x, z, t). \end{aligned} \tag{121.4}$$

122. Geometric presentation of the Lorentz-Einstein transformation. There are many interesting relations between the various physical quantities so far discussed, which may be best shown by a geometric presentation first given by Minkowski. As a preliminary it is necessary to note the fact that in a space of n dimensions in which a point is defined by n rectangular Cartesian coördinates $(x_1, x_2, \cdots, x_n)$, the distance of any point from the origin being given by the square root of the sum of the squares of the coördinates, this sum of squares is the same for all rectangular Cartesian frames having the same origin. The coördinates in any two frames X and Y will be connected with one another by linear equations of the type

$$\begin{aligned} y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n, \\ y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ y_n &= c_{n1}x_1 + c_{n2}x_2 + \cdots + c_{nn}x_n. \end{aligned} \tag{122.1}$$

The necessary and sufficient conditions imposed on the coefficients of this transformation, in order that the sum of squares of the x's, whatever values these may assume, may be equal to the sum of the squares of the corresponding y's, are easily seen to be as follows:

(1) n equations of the type

$$c_{1r}^2 + c_{2r}^2 + c_{3r}^2 + \cdots + c_{nr}^2 = 1. \quad (r = 1, 2, \cdots, n) \tag{122.2}$$

(2) $\dfrac{n(n-1)}{2}$ equations of the type

$$c_{1r}c_{1s} + c_{2r}c_{2s} + \cdots + c_{nr}c_{ns} = 0. \quad (r, s = 1, 2, \cdots, n, r \neq s) \tag{122.3}$$

The transformation (122.1) from the x to the y coördinates is said to be an orthogonal one when the relations (122.2) and (122.3) are satisfied; and we find at once, on multiplying the equations (122.1) by $c_{11}, c_{21}, \cdots, c_{n1}$ etc. in turn, and adding, that

$$\begin{aligned} x_1 &= c_{11}y_1 + c_{21}y_2 + \cdots + c_{n1}y_n, \\ x_2 &= c_{12}y_1 + c_{22}y_2 + \cdots + c_{n2}y_n, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n &= c_{1n}y_1 + c_{2n}y_2 + \cdots + c_{nn}y_n. \end{aligned} \qquad (122.4)$$

It follows from these that since the sum of the squares of the x's is the same as the sum of the squares of the y's, no matter what values the y's may assume, the following relations hold:

(3) $\quad c_{r1}^2 + c_{r2}^2 + \cdots + c_{rn}^2 = 1, \quad (r = 1, 2, \cdots, n) \qquad$ (122.5)

(4) $\quad c_{r1}c_{s1} + c_{r2}c_{s2} + \cdots + c_{rn}c_{sn} = 0. \quad (r, s = 1, 2, \cdots, n,\ r \neq s) \qquad$ (122.6)

These relations are therefore an algebraic consequence of the relations (122.2) and (122.3). Since there are $n + \frac{n(n-1)}{2}$ or $\frac{n(n+1)}{2}$ independent relations on the n^2 coefficients c_{rs}, only $\frac{n(n-1)}{2}$ of these may be varied arbitrarily. In other words, an orthogonal transformation in n dimensions depends on $\frac{n(n-1)}{2}$ parameters.

The equations (119.1) defining the Lorentz-Einstein transformation are linear but not orthogonal, since the relations (122.2) are not satisfied. However, the relation (119.2) suggests that the introduction of the imaginary variable $s = iVt$ will make the transformation an orthogonal one in space of four dimensions. Then (119.2) takes the form

$$x'^2 + y'^2 + z'^2 + s'^2 = x^2 + y^2 + z^2 + s^2. \qquad (122.7)$$

The new form of the transformation is

$$x' = \beta\left(x + i\frac{v}{V}s\right), \quad y' = y, \quad z' = z, \quad s' = \beta\left(s - i\frac{v}{V}x\right), \qquad (122.8)$$

so that the matrix of the coefficients is as follows:

$$\begin{matrix} \beta & 0 & 0 & \frac{i\beta v}{V} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{i\beta v}{V} & 0 & 0 & \beta \end{matrix} \qquad (122.9)$$

It is evident that these coefficients satisfy the conditions of orthogonality (122.2) and (122.3).

This means that in the space K — that is, to observers attached to this frame, — the coördinates of an "event" would be (x, y, z, iVt), and to observers attached to the frame K' the same event would be described by the coördinates (x', y', z', iVt'). The axes of y and y' and of z and z' coincide. Those of x and iVt and of x' and iVt' will therefore be in a common plane, and the equations (122.8) express the fact that the (x', iVt') axes may be obtained from the (x, iVt) axes by rotating them in their plane through an angle whose cosine is β and whose sine is $\frac{i\beta v}{V}$. Systems K' having different values of v will be described by different rotations. In this way we obtain a great many different "time axes," no one of which is privileged over any other. The space of three dimensions which is orthogonal to the time axis may be taken as the Euclidean space of Newton, and it is apparent that time and space have now no absolute meaning. The ideas are inextricably connected, and the stratification of the space-time continuum of four dimensions into a time axis and orthogonal space axes depends on the particular system K' to which the observer is attached.

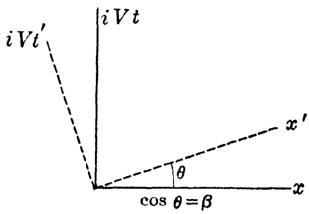

FIG. 39

123. Four-vectors. In exactly the same way as a vector is defined with reference to three orthogonal axes in a space of three dimensions, a four-vector may be defined with reference to orthogonal axes in a space of four dimensions (see Chapter I, § 2). Let the transformation equations connecting the two sets of orthogonal axes have the matrix

$$\begin{array}{ccccc} \overset{\rightarrow}{\downarrow} & x_1 & x_2 & x_3 & x_4 \\ x'_1 & c_{11} & c_{12} & c_{13} & c_{14} \\ x'_2 & c_{21} & c_{22} & c_{23} & c_{24} \\ x'_3 & c_{31} & c_{32} & c_{33} & c_{34} \\ x'_4 & c_{41} & c_{42} & c_{43} & c_{44} \end{array} \qquad (123.1)$$

and let there be a quantity requiring four measurements for its description. Let these, when made by an observer attached to the (x_1, x_2, x_3, x_4) frame, be denoted by (A_1, A_2, A_3, A_4), and when made by an observer attached to the (x'_1, x'_2, x'_3, x'_4) frame be denoted by (A'_1, A'_2, A'_3, A'_4). If these eight numbers are connected by the same relations as the coördinates themselves, that is, if

$$A'_r = c_{r1}A_1 + c_{r2}A_2 + c_{r3}A_3 + c_{r4}A_4, \quad (r = 1, 2, 3, 4) \qquad (123.2)$$

and, consequently,

$$A_r = c_{1r}A'_1 + c_{2r}A'_2 + c_{3r}A'_3 + c_{4r}A'_4, \quad (r = 1, 2, 3, 4), \quad (123.3)$$

then the original quantity is called a four-vector, and (A_1, A_2, A_3, A_4) are called its components with reference to the (x_1, x_2, x_3, x_4) frame. Its components in the other frame are (A'_1, A'_2, A'_3, A'_4). The former represent the aspects of a certain quantity to one observer; the latter, the aspects of the same quantity to a different observer. The order in which the four components are given is essential to the description: A_1, for example, is called the x_1-component, and so on. It follows at once, from the equations (123.2) and the relations (122.2) and (122.3) among the coefficients of an orthogonal transformation of coördinates, that

$$A'^2_1 + A'^2_2 + A'^2_3 + A'^2_4 = A_1^2 + A_2^2 + A_3^2 + A_4^2. \quad (123.4)$$

If this sum is written A^2, the quantity A^2 is called the square of the magnitude of the four-vector. Although the components of a vector in two different frames K and K' are different, the magnitude is the same in both frames. Quantities such as this, which have the same value in all frames, are known as invariant, or scalar, quantities.

If a point (x_1, x_2, x_3, x_4) is so chosen that $x_1 = A_1$, $x_2 = A_2$, $x_3 = A_3$, $x_4 = A_4$, it follows that $x'_1 = A'_1$, $x'_2 = A'_2$, $x'_3 = A'_3$, $x'_4 = A'_4$; and the segment of the straight line joining the origin to the point may be taken as the geometric representation of the vector. Unless we wish to localize the vector, any segment which is equal and parallel to the representative segment just mentioned, and which is similarly directed, would do equally well.

Since any coefficient c_{rs}, say, of the transformation (123.1) has the physical dimensions of the ratio $x'_r : x_s$, the physical dimensions of the components (A_1, A_2, A_3, A_4) must be such that the ratios $A_1 : x_1$, $A_2 : x_2$, $A_3 : x_3$, and $A_4 : x_4$ all have the same dimensions. It follows that the ratios $A'_1 : x'_1$ etc. all have the same physical dimensions.

There are several ways in which four-vectors may be formed.

1. If there are two four-vectors, **A** and **B**, with components (A_1, A_2, A_3, A_4) and (B_1, B_2, B_3, B_4) respectively, then the four quantities $(A_1 + B_1, A_2 + B_2, A_3 + B_3, A_4 + B_4)$ are the components of a four-vector which is called the sum of the two four-vectors **A** and **B** and which is denoted by $\mathbf{A} + \mathbf{B}$. Of course, for this addition to have a physical meaning, **A** and **B** must be of the same kind; that is, the common dimensions of the ratios $A_1 : x_1$, $A_2 : x_2$, etc. must be the same as the common dimensions of the ratios $B_1 : x_1$, $B_2 : x_2$, etc. Then A_r and B_r $(r = 1, 2, 3, 4)$ have the same dimensions and can be added together.

2. If I is an invariant with reference to change of axes, the four quantities (IA_1, IA_2, IA_3, IA_4) are the components of a four-vector if (A_1, A_2, A_3, A_4) are. This four-vector is known as the product of the vector **A** by the invariant I and is denoted by $I\mathbf{A}$. In particular, if I is a pure number, we have numeric multiplication of a four-vector. Thus, if I is negative unity, $(-A_1, -A_2, -A_3, -A_4)$ are the components of the four-vector $-\mathbf{A}$, which is known as the negative of **A**.

3. If I is an invariant operator, (IA_1, IA_2, IA_3, IA_4) are again the components of a four-vector. For example, (A_1, A_2, A_3, A_4) may be functions of a single parameter τ which is the same for all frames. Then $\left(\frac{dA_1}{d\tau}, \frac{dA_2}{d\tau}, \frac{dA_3}{d\tau}, \frac{dA_4}{d\tau}\right)$ are the components of a four-vector which we may denote by $\frac{d\mathbf{A}}{d\tau}$. If the components are functions of more than one invariant parameter, then the differentials (dA_1, dA_2, dA_3, dA_4) are the components of a four-vector which may be denoted by $d\mathbf{A}$.

124. Illustrations of four-vectors. Many interesting illustrations of four-vectors may be obtained in the Minkowski space-time continuum in which the coördinates are (x, y, z, iVt). The orthogonal transformations of coördinates in this space are the modified Lorentz-Einstein transformations (122.8).

1. The quantities $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1,\ iV(t_2 - t_1))$ are the components of a four-vector, showing that space and time intervals are but different aspects of the same thing (the four-vector space-time interval).

2. If we denote the differential with respect to some invariant parameter by the sign δ, $(\delta x, \delta y, \delta z, iV\,\delta t)$ is a four-vector. The square of the magnitude of this four-vector is the invariant quantity $\delta x^2 + \delta y^2 + \delta z^2 - V^2\,\delta t^2$. It is customary to write

$$\delta s^2 = V^2\,\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2. \qquad (124.1)$$

3. We have seen in (120.8) that $\frac{dt}{\alpha} = dt_0$ is an invariant, where $\alpha = \left(1 - \frac{w^2}{V^2}\right)^{-\frac{1}{2}}$. Hence $\frac{d}{dt_0} = \alpha\frac{d}{dt}$ is an invariant operator. Operating on the position-time four-vector (x, y, z, iVt), this invariant operator produces the four-vector

$$\left(\alpha\frac{dx}{dt}, \alpha\frac{dy}{dt}, \alpha\frac{dz}{dt}, i\alpha V\right). \qquad (124.2)$$

This is known as the velocity four-vector, and may be written

$$(\alpha\mathbf{w}, i\alpha V).$$

The square of its magnitude is

$$\alpha^2(w^2 - V^2). \qquad (124.3)$$

Repeating the invariant operation $\alpha\frac{d}{dt}$ on the velocity four-vector (or tensor), we obtain the acceleration tensor

$$\left(\alpha\frac{d}{dt}(\alpha\mathbf{w}), i\alpha V\frac{d\alpha}{dt}\right). \qquad (124.4)$$

Upon multiplication of the acceleration tensor by the invariant rest mass m_0, we obtain a new four-vector which is known as the force tensor. It is

$$\left(m_0\alpha\frac{d}{dt}(\alpha\mathbf{w}), im_0\alpha V\frac{d\alpha}{dt}\right). \qquad (124.5)$$

We have already seen, in (120.15), that the ordinary force vector is $\mathbf{F} = m_0\frac{d}{dt}(\alpha\mathbf{w})$ and, in (120.18), that the activity $(\mathbf{F}.\mathbf{w}) = m_0V^2\frac{d\alpha}{dt}$. Hence the force tensor may be written in the form

$$\left(\alpha\mathbf{F}, \frac{i\alpha}{V}(\mathbf{F}.\mathbf{w})\right), \qquad (124.6)$$

or, introducing the momentum vector **G** and the kinetic energy T, as

$$\left(\alpha\frac{d\mathbf{G}}{dt}, \frac{i\alpha}{V}\frac{dT}{dt}\right). \qquad (124.7)$$

If the velocity tensor (124.2) is multiplied by the invariant rest mass m_0 we get the momentum tensor

$$\left(\mathbf{G},\ \frac{i}{V}(T + m_0V^2)\right) \tag{124.8}$$

(see (120.21)). We thus see how linear momentum and kinetic energy are united in the same tensor.

4. Referring to the equations (121.4), let us see how an observer in the K' system would estimate the magnitude of an element of volume which is at rest in the K system. In calculating the element of volume in the K' system, all measurements will have to be supposed to be made simultaneously in that system. That is, we shall have to regard t' as a constant; and this imposes a certain relation on x and t, so that (x', y', z', t') are now functions of three independent variables (x, y, z) instead of four variables (x, y, z, t) as in the formulas (121.4). We find

$$d(x', y', z') = \beta\, d(x, y, z)\left(1 - v\frac{\partial t}{\partial x}\right),$$

where the fact that t' is constant forces $\frac{\partial t}{\partial x}$ to be $\frac{v}{V^2}$. This gives

$$d(x', y', z') = \frac{d(x, y, z)}{\beta}.$$

Let us now generalize by considering an element of volume which is at rest in neither of the systems K and K'. By introducing an auxiliary system K_0 in which the elementary volume has momentarily the velocity zero, we find

$$d(x, y, z) = \frac{d(x_0, y_0, z_0)}{\alpha},$$

$$d(x', y', z') = \frac{d(x_0, y_0, z_0)}{\alpha'},$$

where α and α' have the values given in (120.8) and (120.9). It follows that

$$\alpha\, d(x, y, z) = \alpha' d(x', y', z') \tag{124.9}$$

(a result which is, indeed, obvious if we consider that elements perpendicular to the direction of motion are not altered). Now we saw in (121.3) that $\rho' = \beta\rho\left(1 - \frac{vw_x}{V^2}\right) = \rho\frac{dt'}{dt}$. Hence, by

(120.8) and (120.9), we have $\frac{\rho'}{\alpha'} = \frac{\rho}{\alpha}$, so that $\frac{\rho}{\alpha}$ is an invariant. It follows, from (124.9), that

$$\rho' d(x', y', z') = \rho\, d(x, y, z), \tag{124.10}$$

a result which is known as the principle of invariance of charge.

Since $\alpha \frac{d}{dt}$ is an invariant operator, it follows that $\rho \frac{d}{dt}$ is an invariant operator. Using it on the velocity tensor, we obtain the four-vector

$$(\rho \mathrm{w},\ i\rho V).$$

Denoting the current density by j, we have $\mathrm{j} = \rho \mathrm{w}$; whence we get the current-density tensor

$$(\mathrm{j},\ i\rho V). \tag{124.11}$$

The square of its magnitude gives the invariant $j^2 - \rho^2 V^2$.

5. In three-dimensional space with orthogonal Cartesian coördinates any plane area may be projected upon the coördinate planes; and if (x, y, z) are the coördinates of any point on the perpendicular from the origin to the plane of the area, we have

$$A_x : A_y : A_z = x : y : z$$

as the ratios of the projected areas. If the area is an elementary one on a surface, we may write

$$d(y, z) : d(z, x) : d(x, y) = x : y : z,$$

and may speak of the vector element of area. Similarly, in space of four dimensions with rectangular Cartesian coördinates we may write

$$d(y, z, s) : d(z, x, s) : d(x, y, s) : -d(x, y, z) = x : y : z : s.$$

The negative sign is due to the fact that we are really erasing, one after another, the columns of a three-row matrix with four columns, and that the signs follow the law of signs for cofactors of a determinant. We may, then, speak of the four-vector element of "volume," and represent it by a segment along the line perpendicular to the orientation determined by the "volume element."

EXERCISE

Rewrite the formulas (121.4), putting $s = iVt$, $s' = iVt'$. Check the fact that the matrix of the transformation from $d(yzs)$, $d(zxs)$, $d(xys)$, $-d(xyz)$ to $d(y'z's')$ etc. is the matrix (122.9) of the transformation from (x, y, z, s) to (x', y', z', s').

125. Tensors of the first order. It may not be out of place to explain why these four-vectors have been called tensors, and, in so doing, to define tensors in general.

Let there be a finite number n of independent variables $(x_1, x_2, \cdots, x_n)$, and let there be given n transformation equations connecting these variables with n others, $(y_1, y_2, \cdots, y_n)$. That is, let

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \cdots, x_n), \\ y_2 &= f_2(x_1, x_2, \cdots, x_n), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n &= f_n(x_1, x_2, \cdots, x_n), \end{aligned} \tag{125.1}$$

where these equations are supposed to be solvable, implying that the Jacobian determinant $\dfrac{\partial(y_1, y_2, \cdots, y_n)}{\partial(x_1, x_2, \cdots, x_n)}$ of the transformation does not vanish. Then, on solving for the x's, we have n equations of the type

$$\begin{aligned} x_1 &= F_1(y_1, y_2, \cdots, y_n), \\ x_2 &= F_2(y_1, y_2, \cdots, y_n), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n &= F_n(y_1, y_2, \cdots, y_n). \end{aligned} \tag{125.2}$$

A point, then, in the n-fold space is defined by either the ordered set $(x_1, x_2, \cdots, x_n)$ or the ordered set $(y_1, y_2, \cdots, y_n)$ of n numbers, and we may refer to x and y coördinate systems.

Furthermore, let there be n given functions of $(x_1, x_2, \cdots, x_n)$ called $(A_1, A_2, \cdots, A_n)$. By the substitution (125.2) these may be made functions of $(y_1, y_2, \cdots, y_n)$. We shall define two processes for the transformation of these functions of $(x_1, x_2, \cdots, x_n)$ into another set of functions of $(y_1, y_2, \cdots, y_n)$.

a. Covariant. Define $(A'_1, A'_2, \cdots, A'_n)$ as functions of $(y_1, y_2, \cdots, y_n)$ given by

$$\begin{aligned} A'_1 &= A_1 \frac{\partial x_1}{\partial y_1} + A_2 \frac{\partial x_2}{\partial y_1} + \cdots + A_n \frac{\partial x_n}{\partial y_1}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ A'_n &= A_1 \frac{\partial x_1}{\partial y_n} + A_2 \frac{\partial x_2}{\partial y_n} + \cdots + A_n \frac{\partial x_n}{\partial y_n}, \end{aligned} \tag{125.3}$$

in which $(A_1, A_2, \cdots, A_n)$ and the coefficients $\dfrac{\partial x}{\partial y}$ are expressed explicitly as functions of $(y_1, y_2, \cdots, y_n)$. It is obvious that this mathematical process cannot have any meaning unless all the sets of products $A_1x_1, A_2x_2, \cdots, A_nx_n$ have the same physical dimensions, that is, involve mass, length, and time to the same degree. Otherwise one could not speak of the *sum* of the terms $A_1\frac{\partial x_1}{\partial y_1}, A_2\frac{\partial x_2}{\partial y_1}, \cdots, A_n\frac{\partial x_n}{\partial y_1}$, etc. If there are, then, n functions of $(x_1, x_2, \cdots, x_n)$, whose dimensions satisfy this condition, n other functions of $(y_1, y_2, \cdots, y_n)$ can be formed by the process defined in (125.3); and it is obvious that the dimensions of these quantities are such that the products $A'_1y_1, A'_2y_2, \cdots, A'_ny_n$ all have the same dimensions. In this case the operation is called a covariant transformation of the first order, and $(A_1, A_2, \cdots, A_n)$ are called the components in the x coördinate system of a covariant tensor of the first rank. Moreover, $(A'_1, A'_2, \cdots, A'_n)$ are the components of the *same* covariant tensor of the first rank in the y coördinate system. These two sets of components represent the n different aspects of a single "quantity" as viewed by two observers having different reference frames.

By convention it is agreed to write the equations (125.3) in the abbreviated form

$$A'_1 = A_\alpha\frac{\partial x_\alpha}{\partial y_1}, \quad A'_2 = A_\alpha\frac{\partial x_\alpha}{\partial y_2}, \quad \cdots,$$

or, in general,

$$A'_r = A_\alpha\frac{\partial x_\alpha}{\partial y_r}, \quad (r = 1, 2, \cdots, n) \tag{125.4}$$

the convention being that the appearance of the same index letter twice in a single term, provided it is written in *Greek*, means that this term represents a sum of n terms in which the letter is put equal, in turn, to $1, 2, \cdots, n$.

As an illustration, let $(A_1, A_2, \cdots, A_n)$ be functions having a potential such that

$$A_1 = \frac{\partial V}{\partial x_1}, \quad A_2 = \frac{\partial V}{\partial x_2}, \quad \cdots, \quad A_n = \frac{\partial V}{\partial x_n}. \tag{125.5}$$

Then

$$A'_1 = \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial y_1} + \cdots + \frac{\partial V}{\partial x_n}\frac{\partial x_n}{\partial y_1} = \frac{\partial V}{\partial y_1},$$

where we suppose V to be expressed in terms of the y's, by means of the formulas (125.2), before differentiation. In general, it turns out that

$$A'_r = \frac{\partial V}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial y_r} = \frac{\partial V}{\partial y_r}. \tag{125.6}$$

Accordingly both sets of components are derived from the *same* potential V. Of course, the form of this potential is different in the two coördinate systems, owing to the fact that the x's are used in one system and the y's in the other; but the value taken at any point is the same, no matter what coördinates are used. If we indicate that the form is changed by putting a prime on V when it is expressed in terms of the y coördinates, equation (125.6) becomes

$$A'_r = \frac{\partial V'}{\partial y_r},$$

to which must be added the equation of definition of V',

$$V' = V. \tag{125.7}$$

A function such as V, which keeps its value at any point, independently of its mode of expression in any particular coördinate system, is called a *scalar* or *invariant* point function. The covariant tensor of the first rank defined by the equations (125.5) is known as the gradient of the scalar point function V; its importance and the obvious mode of transformation of its components may serve to remove some of the apparent artificiality of the transformation (125.3).

b. Contravariant. Starting out once more with n functions $(A_1, A_2, \cdots, A_n)$ of the variables $(x_1, x_2, \cdots, x_n)$, we define n functions $(A'_1, A'_2, \cdots, A'_n)$ of the variables $(y_1, y_2, \cdots, y_n)$ by means of the equations

$$\begin{aligned} A'_1 &= A_1 \frac{\partial y_1}{\partial x_1} + A_2 \frac{\partial y_1}{\partial x_2} + \cdots + A_n \frac{\partial y_1}{\partial x_n}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A'_n &= A_1 \frac{\partial y_n}{\partial x_1} + A_2 \frac{\partial y_n}{\partial x_2} + \cdots + A_n \frac{\partial y_n}{\partial x_n}, \end{aligned} \tag{125.8}$$

where the $(A_1, A_2, \cdots, A_n)$, and the coefficients $\frac{\partial y}{\partial x}$, are supposed to be expressed explicitly in terms of the y's by means of the equations (125.2). It is obvious, as before, that the mathematical process has no physical meaning unless the quantities with which we start out are such that the n ratios $A_1 : x_1, \cdots, A_n : x_n$ have the same physical dimensions. Otherwise the sums used could not be formed. It follows at once that if these ratios have the same physical dimensions, the ratios $A'_1 : y_1, \cdots, A'_n : y_n$ have the same dimensions.

The transformation (125.8) from the n functions A of the x's to the n functions A' of the y's is quite different from the covariant transformation of the first order (125.3). The new type of transformation is distinguished from the covariant type by calling it a contravariant transformation of the first order. Since the set of functions A are of a different kind from those discussed in the previous case, it is proper to emphasize this fact by using different symbols for them; so they are written $(A^1, A^2, \cdots, A^n)$ and $(A'^1, A'^2, \cdots, A'^n)$ respectively. The former are called the components with reference to the x coördinate system of a contravariant tensor of the first rank; the latter, its components with reference to the y coördinate system. Equations (125.8), defining the transformation to which the components of a contravariant tensor of the first rank are subject when we pass from one coördinate system to another, may, in accordance with the convention mentioned above, be written in the form

$$A'^r = A^\alpha \frac{\partial y_r}{\partial x_\alpha}. \quad (r = 1, 2, \cdots, n) \tag{125.9}$$

EXAMPLE

The n differentials $(dx_1, dx_2, \cdots, dx_n)$ may be chosen as the n components in the x coördinate system of a contravariant tensor of the first rank, since the ratios $dx_r : x_r$ all have the same dimensions, namely, zero. The components of this tensor in the y coördinate system are

$$A'^1 = dx_1 \frac{\partial y_1}{\partial x_1} + \cdots + dx_n \frac{\partial y_1}{\partial x_n} = dy_1,$$

and, in general,

$$A'^r = dy_r.$$

Hence the n differentials of the coördinates are the components in each coördinate system of the *same* contravariant tensor of the first rank. This tensor is called the direction tensor. It was this tensor that first suggested the importance of studying the contravariant law (125.8) of transformation.

Remark. The reader will remark that since the differentials of the coördinates form a *contravariant* tensor, the separate components dx_r should have the attached labels above, instead of below, if the notation is to be kept consistent. The usual practice is to indicate the coördinates of a point by the symbols $(x^1, x^2, \cdots, x^n)$, so that the rth component of the direction tensor in the x coördinate system is dx^r. We shall not, however, in this brief introduction follow this convention.

If $(A_1, A_2, \cdots, A_n)$ are the components, in the x coördinate system, of a covariant tensor of the first rank, so that the products x_1A_1, x_2A_2, etc. have the same physical dimensions, the reciprocals $\frac{1}{A_1}$, $\frac{1}{A_2}$, etc. may be chosen as the components in the same coördinate system of a contravariant tensor of the first rank, since the ratios $x_r : \frac{1}{A_r}$ all have the same dimensions. However, the connection between the two tensors, one covariant and the other contravariant, would be quite peculiar to the coördinate system x, as it would not follow that $A'^1 = \frac{1}{A'_1}$ etc. when $A^1 = \frac{1}{A_1}$ etc.

126. Tensors of higher order. Let us suppose that we have n^2 functions of $(x_1, x_2, \cdots, x_n)$, which are arranged in a definite order and which we may denote by

$$\begin{array}{l} A_{11}, A_{12}, \cdots, A_{1n}, \\ A_{21}, A_{22}, \cdots, A_{2n}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_{n1}, A_{n2}, \cdots A_{nn}. \end{array}$$

A covariant transformation of the second order is, by definition, one by which n^2 functions A' of the variables $(y_1, y_2, \cdots, y_n)$ are defined by the conventional formula

$$A'_{rs} = A_{\alpha\beta} \frac{\partial x_\alpha}{\partial y_r} \frac{\partial x_\beta}{\partial y_s}. \quad (r, s = 1, 2, \cdots, n) \qquad (126.1)$$

Thus
$$A'_{34} = A_{11} \frac{\partial x_1}{\partial y_3} \frac{\partial x_1}{\partial y_4} + A_{12} \frac{\partial x_1}{\partial y_3} \frac{\partial x_2}{\partial y_1} + \cdots + A_{21} \frac{\partial x_2}{\partial y_3} \frac{\partial x_1}{\partial y_4} + \cdots + A_{nn} \frac{\partial x_n}{\partial y_3} \frac{\partial x_n}{\partial y_4}.$$

It is, as before, evident that only certain types of functions A_{rs} lend themselves to a transformation of this type, namely, only those groups for which the products $A_{11}x_1x_1$, $A_{12}x_1x_2$, and, in general, $A_{rs}x_rx_s$ have the same physical dimensions. It follows at once that the products $A'_{pq}y_py_q$ all have the same physical dimensions. The functions A_{rs} satisfying this requirement are called the components in the x coördinate system of a covariant tensor of the second rank, and the functions A'_{rs} are the components in the y coördinate system of the *same* tensor.

A contravariant transformation of the second order, on the other hand, is one by which a group of functions A'^{rs} of the n variables y are defined, in terms of a given group of n^2 functions of the n variables x, by means of the formula

$$A'^{rs} = A^{\alpha\beta} \frac{\partial y_r}{\partial x_\alpha} \frac{\partial y_s}{\partial x_\beta}. \quad (r, s = 1, 2, \cdots, n) \qquad (126.2)$$

Thus $A'^{34} = A^{11} \dfrac{\partial y_3}{\partial x_1} \dfrac{\partial y_4}{\partial x_1} + A^{12} \dfrac{\partial y_3}{\partial x_1} \dfrac{\partial y_4}{\partial x_2} + \cdots + A^{nn} \dfrac{\partial y_3}{\partial x_n} \dfrac{\partial y_4}{\partial x_n}.$

Since the formulas (126.2) are quite different from the formulas (126.1) for a covariant transformation, the functions A to which the contravariant operation is applied are distinguished from those to which the covariant transformation is applied by having the attached labels placed as superscripts instead of subscripts. It is evident that the functions A^{rs} must be such that the ratios $A^{rs} : x_r x_s$ (n^2 in number) must all have the same physical dimensions for the indicated mathematical operations in order to have any physical meaning; and it follows as before that if this is so, the ratios $A'^{pq} : y_p y_q$ all have the same dimensions. The n^2 functions A^{rs} are called the components in the x coördinate system of a contravariant tensor of the second rank, and the n^2 functions A'^{pq} are the components in the y coördinate system of the *same* contravariant tensor of the second rank.

Again, if there is a set of n^2 functions $A_s{}^r$ of the variables $(x_1, x_2, \cdots, x_n)$ possessing the property that the quantities $A_q{}^p \dfrac{x_q}{x_p}$, n^2 in number, all have the same physical dimensions, we may apply the mixed transformation

$$A'_s{}^r = A_\beta{}^\alpha \frac{\partial y_r}{\partial x_\alpha} \frac{\partial x_\beta}{\partial y_s} \quad (r, s = 1, 2, \cdots, n) \qquad (126.3)$$

to obtain a new set of n^2 functions of the variables y. The n^2 functions $A_s{}^r$ are, in this case, called the components in the x coördinate system of a mixed tensor of the second rank, partly contravariant and partly covariant.

Similarly, if there is a set of functions A_{rst}, n^3 in number, of the variables x and of the proper dimensions, a covariant transformation of the third order would be defined by

$$A'_{rst} = A_{\alpha\beta\gamma} \frac{\partial x_\alpha}{\partial y_r} \frac{\partial x_\beta}{\partial y_s} \frac{\partial x_\gamma}{\partial y_t}. \quad (r\ s, t = 1, 2, \cdots, n) \quad (126.4)$$

Moreover, contravariant transformations and two types of mixed transformations of the third order may be defined for other sets of functions with different properties. These differences may be indicated by using for the three different sets the symbols

$$A^{mpq}, \quad A_q{}^{mp}, \quad A_{pq}{}^m.$$

The transformations are obvious.

127. Symmetric and skew-symmetric tensors. A symmetric tensor of the second order is one in which $A_{pq} = A_{qp}$. It readily follows, from the equations of transformation, that this implies $A'_{pq} = A'_{qp}$, showing that the property of being symmetric is not peculiar to any particular coördinate system but is, rather, a property of the tensor itself. There are only $\frac{n(n+1)}{2}$ distinct components of a symmetric tensor of the second order.

A skew-symmetric tensor of the second order is one for which $A_{pq} = -A_{qp}$. It is again obvious, from the equations of transformation, that this is properly a quality of the tensor itself, not having anything to do with the particular coördinate system in use. Since here $A_{pp} = -A_{pp}$, the n *diagonal* components vanish; and, the others being equal and opposite in sign in pairs, there are only $\frac{n(n-1)}{2}$ distinct components of a skew-symmetric tensor of the second order. Owing to this fact the transformation equations take a relatively simple form, thus:

$$\begin{aligned} A'_{rs} &= A_{12}\left(\frac{\partial x_1}{\partial y_r}\frac{\partial x_2}{\partial y_s} - \frac{\partial x_2}{\partial y_r}\frac{\partial x_1}{\partial y_s}\right) + A_{13}(\cdots) + \cdots \\ &= A_{\alpha\beta}\frac{\partial(x_\alpha, x_\beta)}{\partial(y_r, y_s)}. \qquad (127.1) \end{aligned}$$

In the summation on the right it is understood that no Jacobian is repeated. Similarly, for a skew-symmetric contravariant tensor of the second rank the equations of transformation run

$$A'^{rs} = A^{\alpha\beta}\frac{\partial(y_r, y_s)}{\partial(x_\alpha, x_\beta)}. \qquad (127.2)$$

Let us suppose that we are considering a space of four dimensions, and let the six distinct components of an alternating or skew-symmetric covariant tensor of the second rank be written in the order $(A_{23}, A_{31}, A_{12}, A_{14}, A_{24}, A_{34})$. Let us confine our attention to orthogonal transformations, and among these orthogonal transformations let us consider for the moment only those which involve merely the variables (x_1, x_2, x_3). Then three of the Jacobians in the summation on the right of (127.1) (those involving the derivatives of x_4, in fact) vanish in the expressions for A'_{23}, A'_{31}, and A'_{12}. We have, for example,

$$A'_{23} = A_{23}\frac{\partial(x_2, x_3)}{\partial(y_2, y_3)} + A_{31}\frac{\partial(x_3, x_1)}{\partial(y_2, y_3)} + A_{12}\frac{\partial(x_1, x_2)}{\partial(y_2, y_3)}.$$

Now the transformation from the three variables (x_1, x_2, x_3) to the three variables (y_1, y_2, y_3) is an orthogonal one, and so the Jacobian $\frac{\partial(x_2, x_3)}{\partial(y_2, y_3)}$, for example, is equal to $\frac{\partial x_1}{\partial y_1}$ (see (3.3)). Hence, if we write $A_{23} = B_1$, etc., our formulas take the form

$$B'_1 = B_1\frac{\partial x_1}{\partial y_1} + B_2\frac{\partial x_2}{\partial y_1} + B_3\frac{\partial x_3}{\partial y_1}, \text{ etc.}$$

This tells us that the three quantities (A_{23}, A_{31}, A_{12}) may be regarded as the components of a vector in the three-dimensional subspace obtained by setting x_4 constant. Similarly, the three quantities (A_{14}, A_{24}, A_{34}) are the three components in the x coördinate system of a vector, provided the fourth variable x_4 does not enter the transformation. Thus the alternating tensor might be written in the form (**B**, **C**), but there is a physical distinction between the vectors **B** and **C** to which attention should be called. The three quantities A_{23}, A_{31}, A_{12} are such that $A_{23}x_2x_3$, $A_{31}x_3x_1$, $A_{12}x_1x_2$ all have the same physical dimensions, and hence the three ratios $B_1 : x_1$, $B_2 : x_2$, $B_3 : x_3$ all have the same physical dimensions. By a similar argument the three products C_1x_1, C_2x_2, C_3x_3 all have the same physical dimensions. In other words, the vector **B** is contravariant in type, and the vector **C** is covariant in type.

A similar analysis of an alternating contravariant tensor of the second rank may be made. The first of the two vectors will now be covariant; the second, contravariant.

A symmetric tensor — covariant, let us say — of the third order is one for which

$$A_{mpq} = A_{pmq} = A_{qpm} = A_{mqp} = A_{pqm} = A_{qmp},$$
$$A_{mmp} = A_{mpm} = A_{pmm}.$$

There are $\frac{n(n-1)(n-2)}{6}$ distinct terms of the former type, and $n(n - 1)$ of the latter. In addition, there are n terms of the type

A_{mmm}. The total number of distinct terms is then $\frac{n(n+1)(n+2)}{6}$, as is evident directly.

A skew-symmetric, or alternating, tensor of the third order is one in which an interchange of two index letters changes the algebraic sign but not the numeric magnitude of the component in question. As before, it is readily apparent that this is a quality of the tensor itself and has nothing to do with the particular coördinate system in which we are examining its components. We have

$$\begin{aligned} A_{mpq} &= -A_{pmq} = A_{qmp} = -A_{mqp} = A_{pqm} = -A_{qpm}, \\ A_{mmp} &= -A_{mmp} = 0, \\ A_{mmm} &= -A_{mmm} = 0. \end{aligned}$$

There are consequently $3\,n(n-1) + n$ or $3\,n^2 - 2\,n$ terms which vanish. The remaining $n^3 - 3\,n^2 + 2\,n$ or $n(n-1)(n-2)$ terms may be grouped in sets of six, three of each six being equal and three being equal to these in magnitude but opposite in sign. There are, accordingly, $\frac{n(n-1)(n-2)}{6}$ distinct terms. Owing to the relations given above, the formulas of transformation take the relatively simple form

$$A'_{rst} = A_{\alpha\beta\gamma}\frac{\partial(x_\alpha, x_\beta, x_\gamma)}{\partial(y_r, y_s, y_t)}, \tag{127.3}$$

where it is understood, as before, that no Jacobian is repeated in the summation.

128. Use of orthogonal axes. The type of transformation from the variables x to the variables y which we have been considering in the foregoing paragraphs has been quite general. All the properties of tensors, however, simplify if the coördinates x and y are orthogonal and Cartesian, so that the transformation is of the type considered in § 123. The transformation from x to y being written in the form

$$y_r = c_{r\alpha}x_\alpha,$$

the transformation from y to x is

$$x_r = c_{\alpha r}y_\alpha$$

(see (123.2) and (123.3)). It follows at once that

$$\frac{\partial x_r}{\partial y_s} = c_{sr} = \frac{\partial y_s}{\partial x_r}, \tag{128.1}$$

showing that there is now no distinction whatever between the covariant type of transformation defined by (125.3) and the contravariant type defined by (125.9).

The coördinates x of a point, when we restrict ourselves to transformations from one set of rectangular Cartesian coördinates to another, are now the components in the x coördinate system of a tensor of the first rank, which is called the position vector. Since the distance of a point from the common origin of the coördinate frames is independent of the particular system of coördinates in use, we have the invariant $x_1^2 + x_2^2 + x_3^2 + x_4^2$. Similarly, from the formulas

$$A'_r = A_\alpha \frac{\partial x_\alpha}{\partial y_r} = c_{r\alpha} A_\alpha \tag{128.2}$$

connecting the components of any tensor of the first rank (or vector) in two coördinate frames, we derive the invariant

$$A_1^2 + A_2^2 + A_3^2 + A_4^2, \tag{128.3}$$

which is known as the square of the magnitude of the tensor. In a four-dimensional space in which transformations from one set of Cartesian axes to another are being considered, a tensor of the first rank is known as a four-vector. Any four-vector may be represented geometrically by the linear segment joining the origin to the point whose coördinates are (A_1, A_2, A_3, A_4).

There is a similar simplification for tensors of higher rank. A covariant tensor of the second rank has its components subject to the transformation

$$A'_{rs} = A_{\alpha\beta} \frac{\partial x_\alpha}{\partial y_r} \frac{\partial x_\beta}{\partial y_s} = c_{r\alpha} c_{s\beta} A_{\alpha\beta},$$

and a contravariant tensor of the second rank has its components subject to the transformation

$$A'^{rs} = A^{\alpha\beta} \frac{\partial y_r}{\partial x_\alpha} \frac{\partial y_s}{\partial x_\beta} = c_{r\alpha} c_{s\beta} A^{\alpha\beta};$$

so that the transformation for covariant and contravariant tensors of the second rank is the same. The alternating tensors of the second rank, in space of four dimensions, are known as six-vectors. We have already seen how to present them as a combination of two vectors in the subspace of three dimensions obtained by not subjecting the fourth variable to any change and not letting it enter the transformation of the other three variables.

129. The products of tensors. It is an immediate consequence of the formulas (125.4) and (126.1) that if the n quantities $(A_1, \cdots, A_n)$ are the components, in any system of coördinates x, of a covariant tensor of the first rank, and if the n quantities $(B_1, \cdots, B_n)$ are the components, in the same system of coördinates, of a second covariant tensor of the first rank, then the n^2 quantities $(A_1B_1, A_1B_2, \cdots, A_nB_n)$, arranged in the order indicated by $A_{rs} = A_rB_s$, are the components, in the x system of coördinates, of a covariant tensor of the second rank. This tensor is called the simple product of the tensors **A** and **B** in the order **AB**. It is at once evident that this kind of product is not commutative in general; that is, it is not in general true that the product **BA** is the same as the product **AB**. In fact, the component B_{rs}, say, of the product **BA** is B_rA_s. If now we form the difference of the two products **AB** and **BA**, we derive a covariant tensor of the second rank whose rs component is $A_rB_s - B_rA_s$. This new tensor is alternating, and is called the outer, or geometric, product of the two covariant tensors of the first rank, **A** and **B**. When $n = 4$, there are six distinct components in the alternating outer product; and when we limit ourselves to orthogonal transformations, the outer product of two four-vectors is a six-vector. When $n = 3$, there are only three distinct components in the outer product; and when we limit ourselves to orthogonal transformations, we see that the outer product of two vectors is again a vector. This is the vector product [**A**.**B**] of Chapter I. In exactly the same way we can form the outer product of two contravariant tensors of the first rank. As an illustration, let us consider a spread S_2 of two dimensions in a space of n dimensions. The points x on S_2 are expressible in terms of two parameters (u_1, u_2); and at any point x of S_2 we have two direction tensors whose components are, respectively, dx_r and δx_r, where $dx_r = \frac{\partial x_r}{\partial u_1} du_1$ and $\delta x_r = \frac{\partial x_r}{\partial u_2} du_2$. The outer product of these two direction tensors is a contravariant tensor of the second rank whose rs component is

$$A^{rs} = \frac{\partial(x_r, x_s)}{\partial(u_1, u_2)} du_1\, du_2. \qquad (129.1)$$

This component is conveniently denoted by $d(x_r, x_s)$. The tensor derived in this way by forming the outer product of the two parametric direction tensors is called the orientation tensor of the element of surface of S_2 at the point in question. In space of four dimensions, when we limit ourselves to orthogonal transformations, this orientation tensor is a six-vector.

In exactly the same way that simple products of covariant tensors and of contravariant tensors of the first rank have been formed, a simple product of a covariant tensor of the first rank and a contravariant tensor of the first rank may be set up. Thus, if we have a covariant tensor **A** and a contravariant tensor **B**, the n^2 quantities A_rB^s arranged in the order indicated by $A_r^s = A_rB^s$ form a mixed tensor of the second rank, partly covariant and partly contravariant. This follows at once from the formulas (125.4), (125.9), and (126.3). In fact, we have

$$A'{}_r{}^s = A'{}_rB'{}^s = A_\alpha \frac{\partial x_\alpha}{\partial y_r} B^\beta \frac{\partial y_s}{\partial x_\beta} = A_\alpha B^\beta \frac{\partial x_\alpha}{\partial y_r}\frac{\partial y_s}{\partial x_\beta} = A_\alpha{}^\beta \frac{\partial x_\alpha}{\partial y_r}\frac{\partial y_s}{\partial x_\beta},$$

showing that the quantities A_r^s are the components of a mixed tensor. The components $A'{}_r{}^r$, n in number, for which the superscript takes the same numeric value as the subscript, are given by the formula

$$A'{}_r{}^r = A_\alpha{}^\beta \frac{\partial x_\alpha}{\partial y_r}\frac{\partial y_r}{\partial x_\beta};$$

and if we add these n components together, we obtain the relation

$$A'{}_\rho{}^\rho = A_\alpha{}^\beta \frac{\partial x_\alpha}{\partial y_\rho}\frac{\partial y_\rho}{\partial x_\beta}.$$

We have a triple summation on the right; and if we sum with respect to ρ first, we see that the result is unity or zero, according as the value assigned to β is or is not the same as the value assigned to α. For the summation $\frac{\partial x_\alpha}{\partial y_\rho}\frac{\partial y_\rho}{\partial x_\beta}$ is nothing but the derivative of x_α with respect to x_β, the y's acting as intermediary variables. We have, then, the relation

$$A'{}_\rho{}^\rho = A_\alpha{}^\alpha, \tag{129.2}$$

which tells us that the sum of the n components of a mixed tensor of the second rank, for which the numeric value assigned the superscript is the same as the numeric value assigned the subscript, has the same value in all coördinate systems. In other words, this sum is an invariant. It is easy to see that the method by which this result was deduced is applicable to obtain other results of a similar nature. Thus, if we have a mixed tensor of rank three $A_{rs}{}^{t}$, covariant of rank two and contravariant of rank one, the sum of the n components for which the numeric value assigned to s is the same as the numeric value assigned to t is a covariant tensor of rank one. Thus from the mixed tensor $A_{rs}{}^{t}$ we derive the covariant tensor $A_r = A_{r\alpha}{}^{\alpha}$, whose rank is less than that of the original tensor by two. Similarly, we can derive another covariant tensor of rank one, $B_r = A_{\alpha r}{}^{\alpha}$. This process, by which we can derive from any mixed tensor one or more tensors whose rank is less than that of the original one by two, is known as the process of contraction. When the mixed tensor is itself derived as the simple product of two tensors, one covariant and the other contravariant, the double operation — that of first finding the simple product and then contracting the resulting mixed tensor — is known as inner multiplication. Thus the inner product of the covariant tensor of the first rank A_r and the contravariant tensor of the first rank B^s is the invariant $A_\alpha B^\alpha$. In Chapter I we have already met this scalar product of two vectors. As we were there limiting ourselves to orthogonal transformations, the distinction between covariant tensors and contravariant tensors did not appear.

The process of forming the simple product is at once seen to be applicable to tensors of higher rank than the first. A single example will suffice. Thus, if we have a covariant tensor of the second rank, A_{rs}, and a contravariant tensor of the third rank, B^{lmn}, the n^5 quantities $A_{rs}B^{lmn}$, arranged in the order indicated by $A_{rs}{}^{lmn} = A_{rs}B^{lmn}$, are the components of a mixed tensor of the fifth rank, covariant of rank two and contravariant of rank three. A first contraction of this mixed tensor yields the mixed tensor of the third rank, $A_{r\alpha}B^{lm\alpha}$, covariant of rank one and contravariant of rank two. A second contraction yields the contravariant tensor of rank one $A_{\beta\alpha}B^{l\beta\alpha}$.

Similarly, the process of forming the outer product can be extended to tensors of higher order. Thus the outer product of three covariant tensors of rank one, A_r, B_s, C_t, is found by first forming the six simple products obtainable by permuting the order of the factors, and then adding or subtracting these tensors according to the rule of signs which holds in the theory of determinants. Thus the outer product of the three covariant tensors of the first rank is a skew-symmetric covariant tensor of rank three, whose *rst* component is the determinant

$$\begin{vmatrix} A_r & A_s & A_t \\ B_r & B_s & B_t \\ C_r & C_s & C_t \end{vmatrix},$$

and similarly for the outer product of three contravariant tensors of the first rank.

For example, let us consider a spread of three dimensions, S_3, in our space of n dimensions. At any point x of this spread we have three parametric directions, the direction tensors being dx_r, ∂x_s, and δx_t, say, where

$$dx_r = \frac{\partial x_r}{\partial u_1}\, du_1, \quad \partial x_s = \frac{\partial x_s}{\partial u_2}\, du_2, \quad \delta x_t = \frac{\partial x_t}{\partial u_3}\, du_3,$$

(u_1, u_2, u_3) being the three independent parameters in terms of which the various points x of the spread S_3 may be expressed. The outer product of these three direction tensors is an alternating contravariant tensor of rank three whose *rst* component is

$$A^{rst} = \frac{\partial(x_r, x_s, x_t)}{\partial(u_1, u_2, u_3)}\, du_1\, du_2\, du_3. \tag{129.3}$$

This component is usually denoted by $d(x_r, x_s, x_t)$, and the tensor may be said to describe the orientation of the element of volume of the spread S_3 at the point x in question. If we are dealing with a space of four dimensions, the outer, or geometric, product of three tensors, either contravariant or covariant, of the first rank has only four distinct components; and if we are limiting ourselves to orthogonal transformations of coördinates, these may be regarded as the components of a four-vector. In the case treated above, where the outer product describes the orientation of the volume element of S_3, the four-vector is perpendicular to this volume element. A little care is necessary in regard to sign in passing over in this way from the alternating tensor of rank $n - 1$ to the vector. What we are really doing is observing that each element of the matrix of the orthogonal transformation is equal to its cofactor in the determinant of the matrix. Thus, when $n = 3$, the three components of the vector are $(A_{23}, -A_{13}, A_{12})$. If we wish to avoid writing in the negative sign, we may avail ourselves of the alternating character of the tensor of rank two to write these components as (A_{23}, A_{31}, A_{12}). In the case where $n = 4$, the four components of the four-vector are $(A_{234}, -A_{134}, A_{124}, -A_{123})$. If we wish to avoid the minus signs, we may write these $(A_{234}, A_{314}, A_{124}, A_{321})$; but then especial attention must be directed to the reversal in the order of the subscripts in the last term.

EXAMPLES

1. The simple product of A_{rs}, a covariant tensor of rank two, and A^t, a contravariant tensor of rank one, is the mixed tensor of rank three $A_{rs}A^t$. Contracting this in one of the two possible ways, we have one of the inner products of the two tensors; for example, $A_{\alpha s}A^{\alpha} = A_s$. The inner product of this and a second contravariant tensor of rank one, B^r, yields the invariant

$$A_{\beta}B^{\beta}, \quad \text{or} \quad A_{\alpha\beta}A^{\alpha}B^{\beta}.$$

Conversely, if we are assured of the invariancy of this double summation when the quantities A^r and B^s are the components of arbitrary vectors, we may infer that the n^2 quantities A_{rs} are the components of a covariant tensor of rank two.

2. The Lorentz-Einstein transformation, when put in the modified form introduced by Minkowski, is a special orthogonal transformation of four variables. The matrix of the coefficients has been given in (122.9), and from this matrix the components of a four-vector in one system can be read off when the components in the other system are known. As examples of four-vectors, we have already given the velocity tensor $(\alpha\mathbf{w}, i\alpha V)$ and the force tensor $\left(\alpha\mathbf{F}, \frac{i\alpha}{V}(\mathbf{F}.\mathbf{w})\right)$. The matrix from which we can read off the components of a six-vector in one system when its components in the other system are known has six rows and six columns. The number in the pq row and rs column of this matrix of order six is the two-row determinant $\begin{vmatrix} c_{pr} & c_{ps} \\ c_{qr} & c_{qs} \end{vmatrix}$ formed by taking the rows p and q and the columns r and s of the matrix (123.1). In this way we obtain the following matrix, showing how the components of a six-vector are transformed:

$$\begin{array}{ccccccc}
\overset{\rightarrow}{\downarrow} & 23 & 31 & 12 & 14 & 24 & 34 \\
2'3' & 1 & 0 & 0 & 0 & 0 & 0 \\
3'1' & 0 & \beta & 0 & 0 & 0 & \dfrac{i\beta v}{V} \\
1'2' & 0 & 0 & \beta & 0 & -\dfrac{i\beta v}{V} & 0 \\
1'4' & 0 & 0 & 0 & 1 & 0 & 0 \\
2'4' & 0 & 0 & \dfrac{i\beta v}{V} & 0 & \beta & 0 \\
3'4' & 0 & -\dfrac{i\beta v}{V} & 0 & 0 & 0 & \beta
\end{array} \tag{129.4}$$

From this table we read off

$$A'_{23} = A_{23}, \qquad A'_{14} = A_{14},$$

$$A'_{31} = \beta\left(A_{31} + \frac{iv}{V} A_{34}\right), \qquad A'_{24} = \beta\left(A_{24} + \frac{iv}{V} A_{12}\right),$$

$$A'_{12} = \beta\left(A_{12} - \frac{iv}{V} A_{24}\right), \qquad A'_{34} = \beta\left(A_{34} - \frac{iv}{V} A_{31}\right).$$

Upon comparing these equations with equations (121.1), giving the transformation undergone by the electric and magnetic force components when we pass from a system K to a system K' having a uniform velocity v in the direction of the common x-axis relative to K, we see that under a Lorentz transformation the six quantities $\left(E_x, E_y, E_z, \frac{i\mu V}{c} H_x, \frac{i\mu V}{c} H_y, \frac{i\mu V}{c} H_z\right)$ transform like the components of a six-vector. On using the relation $c = V(\epsilon\mu)^{\frac{1}{2}}$, we see that under a Lorentz transformation the six quantities $\left(\mathbf{E}, i\sqrt{\frac{\mu}{\epsilon}}\mathbf{H}\right)$ transform like the components of a six-vector. This is not sufficient to show that these six quantities are actually the components of a six-vector, since the Lorentz transformation is a special orthogonal transformation. But if we remember that there is no privileged direction in our three-dimensional space, we can consider all possible Lorentz transformations which are obtained by varying arbitrarily the direction of the x-axis and the magnitude of the relative velocity v; and in this way all possible orthogonal transformations of the four variables are obtained, showing that $\left(\mathbf{E}, i\sqrt{\frac{\mu}{\epsilon}}\mathbf{H}\right)$ is actually a six-vector. Obvious modifications are obtained by using the definitions $\mathbf{D} = \frac{\epsilon\mathbf{F}}{4\pi}$, $\mathbf{B} = \mu\mathbf{H}$, and remembering that $V = \frac{c}{(\epsilon\mu)^{\frac{1}{2}}}$. Thus there are the six-vectors

$$(\sqrt{\epsilon}\mathbf{E}, i\sqrt{\mu}\mathbf{H}), \left(\frac{c}{V}\mathbf{E}, i\mathbf{B}\right), \left(\mathbf{D}, \frac{c}{4\pi V}\mathbf{H}\right). \qquad (129.5)$$

It is easily verifiable that if (**A**, **B**) constitutes a six-vector so also does (**B**, **A**), and that $A^2 + B^2$ and $(\mathbf{A}.\mathbf{B})$ are invariants. Using these results, a number of invariants of the electromagnetic field can be obtained from (129.5).

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CHAPTER XIX

PHYSICAL DIMENSIONS

130. Definition of dimensions. Illustrations. The fundamental quantities discussed in mechanics are mass, length, and time, and all the physical quantities considered in mechanics are defined in terms of these. Thus the numeric value of a velocity is $\frac{dr}{dt}$; of a force, $m\frac{dv}{dt}$; of kinetic energy, $\frac{1}{2}mv^2$; and so on. Consequently velocity is said to have the dimensions of LT^{-1}; force, those of MLT^{-2}; kinetic energy, those of ML^2T^{-2}; and so on. One cannot add or subtract the numeric measures of two quantities unless these two quantities have the same dimensions. It is meaningless to add the measure of a mass to the measure of a velocity. Any equation in physics, therefore, must involve only terms of the same dimensions. It can then express the equality of two numbers which represent the values of the terms as obtained, either directly or indirectly, by measurements. Further, if vectors appear in an equation, the vector on the right-hand side of the equation must have the same vector sense as the vector on the left. A vector, for instance, in the direction of the x-axis cannot be equated to one in the direction of the y-axis. We shall use rectangular Cartesian axes; and, in order to keep the direction concept in our equations, we shall use the symbols L_x, L_y, L_z to denote lengths measured parallel to the (x, y, z) axes respectively. Thus, instead of saying merely that velocity has the dimensions LT^{-1}, we shall say that it has the dimensions $(L_xT^{-1}, L_yT^{-1}, L_zT^{-1})$, implying thereby that it is a vector and that its three components have, respectively, the dimensions mentioned. Our remark that a vector in the direction of the x-axis cannot be equated to one in the direction of the y-axis implies that a term with dimensions L_xT^{-1}, for example, cannot be equated to a term having dimensions L_yT^{-1}. Every equation involving physical quantities must be homogeneous with respect to M, T, L_x, L_y, L_z.

It is important, therefore, to determine what are the dimensions of the more usual physical quantities which occur in mechanics. The dimensions of the most important of these quantities are given below.

Velocity components $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$: L_xT^{-1}, L_yT^{-1}, L_zT^{-1}.

Momentum components $\left(m\frac{dx}{dt}, m\frac{dy}{dt}, m\frac{dz}{dt}\right)$: ML_xT^{-1}, ML_yT^{-1}, ML_zT^{-1}.

Acceleration components $\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$: L_xT^{-2}, L_yT^{-2}, L_zT^{-2}.

Force components $\left(m\frac{d^2x}{dt^2} \text{ etc.}\right)$: ML_xT^{-2} etc.

Element of area $(d(y, z), d(z, x), d(x, y))$: L_yL_z, L_zL_x, L_xL_y.

Element of volume $(d(x, y, z))$: $L_xL_yL_z$.

Density $\left(\frac{dm}{d\tau}\right)$: $ML_x^{-1}L_y^{-1}L_z^{-1}$.

Work $((\mathbf{F}.\mathbf{ds})$, the two vectors having the same direction): $ML_x^2T^{-2}$, $ML_y^2T^{-2}$, or $ML_z^2T^{-2}$.

Energy $(\frac{1}{2}mv^2)$: $ML_x^2T^{-2}$, $ML_y^2T^{-2}$, or $ML_z^2T^{-2}$.

Activity or power $(\mathbf{F}.\mathbf{v})$: $ML_x^2T^{-3}$, $ML_y^2T^{-3}$, or $ML_z^2T^{-3}$.

Action $\left(m\frac{d\mathbf{r}}{dt}\cdot d\mathbf{r}\right)$: $ML_x^2T^{-1}$, $ML_y^2T^{-1}$, or $ML_z^2T^{-1}$.

Pressure (force per unit area, the area being perpendicular to the force): $ML_xL_y^{-1}L_z^{-1}T^{-2}$, $ML_yL_z^{-1}L_x^{-1}T^{-2}$, etc.

Tangential or shearing stress (force per unit area, the area being parallel to the force): $ML_xL_x^{-1}L_y^{-1}T^{-2}$, that is, $ML_y^{-1}T^{-2}$ etc.

Strain $\left(\frac{d\tau}{\tau}, \frac{dL}{L}, \text{ etc.}\right)$: dimensions are zero.

Coefficient of elasticity, $\frac{\text{stress}}{\text{strain}}$ (that is, Young's modulus): $ML_xL_y^{-1}L_z^{-1}T^{-2}$.

Compressibility, the reciprocal of the coefficient of elasticity, has the dimensions of the reciprocal of pressure: $M^{-1}L_x^{-1}L_yL_zT^2$ etc.

Coefficient of viscosity (tangential stress divided by the change, per unit length perpendicular to the stress, of the velocity parallel to the stress): $\frac{ML_y^{-1}T^{-2}}{L_xT^{-1}L_z^{-1}} = ML_x^{-1}L_y^{-1}L_zT^{-1}$ etc. (The dimensions given correspond to the case where the area is perpendicular to the z-axis.)

Plane angle $\left(\frac{ds}{r}\right.$, where ds is perpendicular to $\left.r\right)$: for instance,

$L_yL_x^{-1}$, $L_xL_y^{-1}$, $L_xL_z^{-1}$, etc.

Solid angle $\left(\frac{dS}{r^2}\right.$, where dS is perpendicular to $\left.r\right)$: $L_x^{-2}L_yL_z$ etc.

Moment of inertia, simplest case (mr^2): ML_x^2 etc. (see below).

Product of inertia: ML_yL_z etc.

Angular velocity: $L_yL_x^{-1}T^{-1}$ etc.

Angular momentum: $ML_xL_yT^{-1}$ etc.

Moment of force: $ML_xL_yT^{-2}$ etc.

Now the formula for the distance between two points in a Euclidean space, when rectangular Cartesian coördinates are employed, is of the type

$$d^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2; \qquad (130.1)$$

and if we are going to employ this formula, we shall have to admit that a quantity of dimensions L_x^2 can be added to a quantity of dimensions L_y^2 or of dimensions L_z^2. In other words, the square of L_x, of L_y, or of L_z has lost the element of direction and is simply the square of a length. (This is illustrated by the dimensions of energy, activity, and action.) For example, $L_xL_yL_z^{-1} = \frac{L_xL_yL_z}{L^2} = L_xL_y^{-1}L_z$, so that $L_yL_z^{-1} = L_y^{-1}L_z$. Instead of doing this, we may regard the unit coefficients of the fundamental formula (130.1) as themselves dimensional quantities, the coefficient of $(\Delta x)^2$ having then, necessarily, the dimensions $\frac{L^2}{L_x^2}$, and likewise for the other coefficients. As we shall wish to add lengths in different directions, we shall as before have to regard L^2 as having lost the element of direction, so that the two points of view are equivalent. However, the latter has the advantage that it enables us to avail ourselves of generalized coördinates in which the distance formula is of the type $d^2 = g_{\alpha\beta}\Delta q_\alpha \Delta q_\beta$. The dimensions of g_{rs} would then be $\frac{L^2}{Q_rQ_s}$ where Q_r, for instance, denotes the dimensions of the generalized coördinate q_r. Directly connected with the distance formula is the identical relation connecting the direction cosines of a line; and a consideration of this formula also shows that L_x^2, L_y^2, L_z^2 must be regarded as having the common dimensions L^2.

It should be noted that there are two distinct groups of vector quantities, which are distinguished by their dimensions. In one group belong vectors such as velocity, force, and acceleration, which have the property that the dimensions of the component with reference to any axis are proportional to the element of length in that direction; so that, calling the vector **A**, and indicating dimensions by the use of a square bracket,

$$[A_x] : [A_y] : [A_z] = L_x : L_y : L_z.$$

The dimensions of $\frac{A_x}{x}, \frac{A_y}{y}, \frac{A_z}{z}$ are all the same. This group contains the vectors which in the preceding chapter were called contravariant tensors of the first rank. In the other group are contained vectors such as angular momentum and moment of force. The vectors of this group are such that the dimensions of the component with reference to any axis are proportional to the products of the elements of the other two axes. Denoting the vector by **B**, we have

$$[B_x] : [B_y] : [B_z] = L_yL_z : L_zL_x : L_xL_y.$$

The dimensions of xB_x, yB_y, zB_z are therefore all the same. This group consists of the covariant vectors of the first rank. As has been seen in the preceding chapter, the outer product of any two vectors of the first group belongs to the second group.

The equation of the momental ellipsoid is

$$Ax^2 + By^2 + Cz^2 - 2\,Fyz - 2\,Gzx - 2\,Hxy = 1,$$

where $A = \Sigma m(y^2 + z^2)$, $B = \Sigma m(z^2 + x^2)$, $C = \Sigma m(x^2 + y^2)$,

$$F = \Sigma myz, \quad G = \Sigma mzx, \quad H = \Sigma mxy.$$

These, when a minus sign is attached to F, G, and H, are the six distinct components of a symmetric covariant tensor of the second rank. In general, the dimensions of the various components A_{xx} etc. of any covariant tensor of rank two are such that the products $A_{xx}x^2, \cdots, A_{yz}yz, \cdots$ all have the same dimensions.

The case of pressure also calls for attention. Writing the components of the normal and tangential stresses at a point as $P_{xx}, P_{yy}, P_{zz}, P_{yz}, P_{zx}, P_{xy}$, their dimensions are $ML_xL_y^{-1}L_z^{-1}T^{-2}$, $ML_yL_z^{-1}L_x^{-1}T^{-2}$, $ML_zL_x^{-1}L_y^{-1}T^{-2}$, $ML_x^{-1}T^{-2}$, $ML_y^{-1}T^{-2}$, $ML_z^{-1}T^{-2}$, respectively. The dimensions are such that the quantities $\frac{P_{xx}}{x^2}, \frac{P_{yy}}{y^2}, \frac{P_{zz}}{z^2}, \frac{P_{yz}}{yz}, \frac{P_{zx}}{zx}, \frac{P_{xy}}{xy}$ all have the same dimensions. The stress tensor is therefore a symmetric contravariant tensor of rank two.

131. Other fundamental dimensions. It has been proposed to adopt, in place of one of the fundamental quantities M, L, T, the concept energy. This is possible, as $E = ML^2T^{-2}$ may be substituted in any one of the formulas.

In other branches of physics than mechanics, quantities enter the formulas for which the expressions in terms of mass, length, and time are not known. Thus the experimental law for the equation of state of a perfect gas is

$$pV = mR\theta,$$

where p is the pressure, V the volume, m the mass, θ the absolute temperature, and R a constant for any one gas. Writing the dimensional equation

$$ML_xL_y^{-1}L_z^{-1}T^{-2}.L_xL_yL_z = M[R\theta],$$

we find

$$[R\theta] = L_x^2T^{-2} = L^2T^{-2}.$$

If R is replaced in the formula for the equation of state by $\frac{R_0}{m'}$, where m' is the mass of a molecule of the gas, R_0 will be a universal constant, the same for all gases, and

$$[R_0\theta] = ML^2T^{-2}.$$

Hence the product $R_0\theta$ has the dimensions of energy, and this is all that can be said.

Similarly, by the equation of definition of entropy S, $dS = \frac{\Delta H}{\theta}$, in which ΔH is the heat added to the body at temperature θ. Hence $S\theta$ has the dimensions of energy, so that R_0 and S have the same dimensions.

Another illustration is furnished by the experimental law for the force between two electric charges e and e', at a distance r apart, in a definite dielectric, namely,

$$\mathbf{F} = \frac{ee'}{\epsilon r^3}\mathbf{r},$$

where ϵ is a constant for any one medium. Writing down the dimensions, we have

$$ML_xT^{-2} = \left[\frac{e^2}{\epsilon}\right]L^{-3}L_x,$$

or

$$\left[\frac{e^2}{\epsilon}\right] = ML^3T^{-2}; \quad \text{whence} \quad [e] = [\epsilon]^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}.$$

Similarly, writing μ as another constant for a medium, depending upon its magnetic properties, the dimensions of a magnetic charge are

$$[m] = [\mu]^{\frac{1}{2}} M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}.$$

Furthermore, since the velocity of electromagnetic waves is $\frac{c}{(\epsilon\mu)^{\frac{1}{2}}}$, where c is a dimensionless number,

$$[\epsilon\mu] = L^{-2}T^{2}.$$

Nothing further is known concerning the dimensions of e, m, ϵ, and μ.

132. Dimensional equations. The properties of dimensions may be used to determine physical laws, provided there is sufficient knowledge of the elements involved in the phenomena. Two simple illustrations will be given; but for a detailed treatment, reference should be made to Bridgman's "Dimensional Analysis" and to papers by Buckingham and others.

Let there be a drop of liquid oscillating under the influence of its surface tension. It will be assumed that the only properties of the drop which affect the period of its vibrations are its density ρ, its volume V, and its surface tension H. Under these conditions the dimensions of the period must be the same as those of some function of ρ, V, and H. Assuming that this function is a simple product of powers of these quantities, we have

$$T = [\rho]^{\alpha}[V]^{\beta}[H]^{\gamma},$$

where (α, β, γ) are unknown exponents which we proceed to determine. Writing in the dimensions of the various quantities involved, we have

$$T = (ML^{-3})^{\alpha}(L)^{3\beta}(MT^{-2})^{\gamma} = M^{\alpha+\gamma}L^{3(\beta-\alpha)}T^{-2\gamma}.$$

This gives the three equations

$$\alpha + \gamma = 0,$$
$$\beta - \alpha = 0,$$
$$-2\gamma = 1;$$

whence we obtain $\gamma = -\frac{1}{2}$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$.

Accordingly, the law of vibration of the drop is that the period T is proportional to $\rho^{\frac{1}{2}}V^{\frac{1}{2}}H^{-\frac{1}{2}}$.

This process can be applied if there are only three variables upon which the period depends. But it is obvious that the viscosity and the coefficient of elasticity may be involved. The viscosity μ, being the ratio of force per unit area to velocity per unit length, has the dimensions of $ML^{-1}T^{-1}$; and, instead of expressing the elasticity explicitly, the velocity of compressional waves, c, in the liquid may be used, the dimensions of which are LT^{-1}. It is then obvious that the dimensions of the ratios $\frac{\mu c}{H}$, $\frac{\mu}{\rho c V^{\frac{1}{3}}}$, and $\frac{H}{\rho c^2 V^{\frac{1}{3}}}$ reduce to zero. Instead of the five variables ρ, V, H, μ, and c, it is necessary, then, to choose three, and two functions of a dimensionless character. Choosing ρ, V, H, and the first two dimensionless ratios just given, our dimensional equation becomes

$$T = [\rho]^{\alpha}[V]^{\beta}[H]^{\gamma} f_1\left(\frac{\mu c}{H}\right) f_2\left(\frac{\mu}{\rho c V^{\frac{1}{3}}}\right).$$

Since the arguments of the functions f_1 and f_2 are dimensionless, we find the same values for α, β, γ as before; and we have

$$T = \rho^{\frac{1}{2}} V^{\frac{1}{2}} H^{-\frac{1}{2}} f_1\left(\frac{\mu c}{H}\right) f_2\left(\frac{\mu}{\rho c V^{\frac{1}{3}}}\right).$$

If we had chosen μ, ρ, and H as the three dimensional quantities, and $\frac{\mu c}{H}$ and $\frac{H}{\rho c^2 V^{\frac{1}{3}}}$ as the two dimensionless ratios, we should have found in precisely the same way

$$T = \mu^3 \rho^{-1} H^{-2} F_1\left(\frac{\mu c}{H}\right) F_2\left(\frac{H}{\rho c^2 V^{\frac{1}{3}}}\right).$$

Since V appears only in the last factor, a further conclusion can be drawn from this last equation, namely, that if experiments are performed upon different drops for which $\frac{H}{\rho c^2 V^{\frac{1}{3}}}$ is the same, the ratio of the periods is independent of the volume.

Another illustration is afforded by the problem of the resistance of a fluid to a sphere moving through it. The possible variables (of which there are five) may be selected, *a priori*, as the density of the fluid, ρ; the linear dimensions of the sphere, l; its velocity, v; the viscosity of the fluid, μ; and the velocity of compressional waves, c. The dimensions of the quantities

$\frac{\rho vl}{\mu}$ and $\frac{v}{c}$ are zero. Hence, choosing μ, ρ, and v as variables, the law of resistance is

$$F = \mu^{\alpha}\rho^{\beta}v^{\gamma}f_1\left(\frac{\rho vl}{\mu}\right)f_2\left(\frac{v}{c}\right);$$

whence $\qquad MLT^{-2} = M^{\alpha}L^{-\alpha}T^{-\alpha}M^{\beta}L^{-3\beta}L^{\gamma}T^{-\gamma}.$

It follows that $\qquad \alpha = 2, \quad \beta = -1, \quad \gamma = 0,$

or $$F = \mu^2\rho^{-1}f_1\left(\frac{\rho vl}{\mu}\right)f_2\left(\frac{v}{c}\right).$$

Experiments show that F is proportional to l^2 over certain ranges of v. Hence we may put $f_1\left(\frac{\rho vl}{\mu}\right) = \left(\frac{\rho vl}{\mu}\right)^2$, and our formula becomes

$$F = \rho v^2l^2f_2\left(\frac{v}{c}\right).$$

In any case, it is clear that if in experiments on two spheres the quantity $\left(\frac{\rho vl}{\mu}\right)$ is the same, the ratio of the forces of resistance is independent of the dimensions of the sphere; that is, that there is no scale effect.

We might have selected ρ, l, and v as variables, so that

$$F = \rho^{\alpha}l^{\beta}v^{\gamma}f_1\left(\frac{\rho vl}{\mu}\right)f_2\left(\frac{v}{c}\right).$$

We find, exactly as before, on equating exponents in the dimensional equation, that

$$\alpha = 1, \quad \beta = 2, \quad \gamma = 2;$$

so that $$F = \rho l^2v^2f_1\left(\frac{\rho vl}{\mu}\right)f_2\left(\frac{v}{c}\right).$$

The expression $\frac{\rho vl}{\mu}$ is known as Reynolds's number.

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